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Fuzzy Gaussian GARCH and Fuzzy Gaussian EGARCH Models: Foreign Exchange Market Forecast

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Abstract

This article discusses a comparison of the GARCH and EGARCH conditional variance methods, with respect to the Fuzzy Gaussian GARCH and Fuzzy Gaussian EGARCH. The returns of four exchange rates were forecasted at daily periodicity from January 2015 to November 2022 and out-of-sample, January 2019, and December 2022. The results indicate that the Fuzzy GARCH and Fuzzy EGARCH models better estimate the volatility behaviour of the exchange market series compared to traditional techniques. Therefore, the recommendation is to estimate other high volatility variables to verify the efficiency of the fuzzy techniques, however, the main limitation is that it is not possible to apply traditional econometric tests for fuzzy techniques because the parameters are not estimated with the logarithm of maximum likelihood. The estimation of the parameters of GARCH and EGARCH models with fuzzy theory is the originality proposal. In conclusion, fuzzy methodologies have less error in forecasting the volatility of in-sample and out-of-sample exchange rates.
JEL Classification: C22, C51, C53.

Keywords: Fuzzy Logic, GARCH, EGARCH, FUZZY GARCH, FUZZY EGARCH.

Modelos GARCH Gaussiano Difuso y EGARCH Gaussiano Difuso: Pronóstico del Mercado Cambiario

Resumen

El presente artículo compara los métodos de varianza condicional GARCH y EGARCH, con respecto a la propuesta Fuzzy Gaussian GARCH y Fuzzy Gaussian EGARCH. Se pronosticó la rentabilidad de cuatro tipos de cambio en periodicidad diaria desde enero 2015 a noviembre 2022 y fuera de muestra, enero 2019 y diciembre 2022. Los resultados revelan que los modelos Fuzzy GARCH y Fuzzy EGARCH estiman mejor el comportamiento de la volatilidad de las series del mercado cambiario en comparación con las técnicas tradicionales. Por lo que, la recomendación es estimar otras variables de alta volatilidad para verificar la eficiencia de las técnicas difusas, sin embargo, la principal limitación es que no es posible aplicar las pruebas econométricas tradicionales para técnicas difusas porque los parámetros no son estimados con el logaritmo de máxima verosimilitud. La estimación de los parámetros de los modelos GARCH y EGARCH con teoría difusa es la propuesta de originalidad. En conclusión, las metodologías difusas tienen menos error al pronosticar la volatilidad de los tipos de cambio dentro muestra y fuera de muestra.
Clasificación JEL: C22, C51, C53.

Palabras clave: Lógica Difusa; GARCH, EGARCH, FUZZY GARCH, FUZZY EGARCH.

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1. Introduction

The important increase in volatility in the markets causes that the forecast of the foreign exchange market variables be increasingly complicated and at the same time is essential for taking decisions. In this way, the models that try to predict these variables need more level of specialty. There are great quantities of analysis tools that allow understand and forecast this series; however, they have not been able to give accurate estimates. Therefore, it makes a study of econometric models based on fuzzy logic and its application to develop forecasts. The principal argument of this research is that the Fuzzy Gaussian GARCH and Fuzzy Gaussian EGARCH models provide better and more accurate forecasts in comparison with the traditional models GARCH and EGARCH.

Fuzzy Logic has its origins in the works of Zadeh (1965), who start from the fact of defining the concept of belonging to a certain set; this gave us how-to consequence the Study of Fuzzy Sets. An important aspect developed by this author is the assignment levels of membership to an element x in a set A, this is called how a membership function; this associates each point in X with a value in the interval [0, 1] of the real numbers.

Furthermore, this study constitutes a generalization of classical logic, since if x is defined in A; it can only take the value of 0 or 1 (belong or not to the set). According to Zadeh (1965), a fuzzy set is a class of objects with a *continuum* that shows their degree of membership. On the other hand, in its analysis, the concept of convexity for fuzzy sets has the same importance that the optimization of classical sets. Fuzzy logic is a multi-valued form of logic, it can take approximate reasoning. For this reason, linguistic variables are used in the definition of sets (Coyaso *et al.*, 2015).

Tanaka *et al.* (1982), for the first time, developed a description of this topic applied to linear regression. The main result of this investigation was the construction, from the Fuzzy Linear Function and the Linear Regression Model, a new model of the econometric analysis that is known as the Fuzzy Linear Regression Model. This methodology is the basis for the development of various similar models about fuzzy regression, how Kim *et al.* (1998), Tanaka (1987), Chang (1997), Cheng *et al*. (1999a), Cheng *et al*. (1999b), Özelkan *et al*. (2000), Dunyak *et al*. (2000).

Another application of fuzzy logic was developed by Song and Chissom (1993a), who studied of fuzzy sets and their applications to decision making, dynamic processes in which observations are linguistic values. These are called Fuzzy Time Series, defining two types of series: time-invariant and time-variant.

Likewise, Song and Chissom (1993b) under the previous definition develop an application of this time-invariant methodology to model the enrollments of the University of Alabama, comparing it with other existing methods for forecast this series. So the advantage of this methodology is that it works in competitive scenarios, other works that start from the contributions of Song and Chissom are Chen (1996), Chen *et al.* (2004), Yu (2005a), Yu (2005b), Huarng (2005).

Tseng *et al.* (2001) from the ARIMA time series model and the fuzzy regression model develops a new methodology called Fuzzy-ARIMA model. This applies to forecast the exchange rate of NT dollars to US dollars. The importance of this model is that provide the decision-makers the best and worst possible situations.

In another way Popov *et al*. (2005), it is known that it presented a new perspective of analysis of the Fuzzy Time Series. Who explained how to integrate the characteristics of Fuzzy Theory

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in the GARCH Models; it generated the model Fuzzy generalization of Autoregressive Conditional Heteroscedasticity (Fuzzy GARCH). It was found that this model generates better results than the basic GARCH model, especially to measure volatility in Financial Time Series. Under the same perspective Hung (2009), developed a model like called Threshold Asymmetric Generalized Autoregressive Conditional Heteroskedasticity Model. The principal result in this research was that the financial series studied with these models indicate that the propagation of volatility in financial markets is non-linear. Although, the method proposed by Hung is still extremely limited to estimate high volatility behaviours.

Other ways to analyse the Fuzzy Time Series was reviewed by Singh (2017), where the main characteristics of the investigations are: determination of the length of intervals, the establishment of fuzzy logical relationships between several factors, and defuzzification in a Hybridize modelling. It also makes a review of the works that have been developed with this methodology.

Dash *et al* (2016), develops a new methodology where it takes the volatility of financial series as a Fuzzy process. In the EGARCH model, its structure of the variance equation is evaluated as a Gaussian membership function and with a Neural network that learns of the Gaussian volatility let generate a forecast. It was found that this hybrid model improves significantly with respect to or models of less complexity.

The structure of this paper is organized as follows: In section 2, Concepts of fuzzy theory, GARCH, and EGARCH models are reviewed. In sections 3 and 4, FUZZY GARCH and FUZZY EGARCH Models with Gaussian Parameters are formulated and proposed. The models are applied to forecasting the foreign exchange rate of (MX peso versus US dollar, Euro versus Pound Sterling, Frank Switzerland versus US dollar and Yen versus Euro) and compared to other time series models in section 5 and finally, the conclusions are discussed.

2. Concepts of Fuzzy Theory, GARCH, and EGARCH models

This section aims to develop the main concepts of fuzzy theory that impact the study of fuzzy time series. As well as the incorporation of aspects is considered relevant for the present study. Returned to the concepts of Zadeh (1965) can be studied and address several definitions that allow developing in this analysis. Therefore, the theoretical study begins with a series of definitions.

Definition. - 1 Let X a space of points, with a generic element of X denoted by x. So, $X = x$. A fuzzy set A in X is characterized by a membership function $\mu_A(x)$, which associates each element of X with one and only one element of the real numbers in the interval [0,1]. Where, the value of $\mu_A(x)$ evaluated in x represents the degree of belonging of x in A.

$$A = \frac{\int_i^{\infty} \mu_A\,(x_i)}{x_i}$$

From Definition 1, it is identified that two types of logic are developed from these types of sets. The first is when the x element can only take values from the end of the interval. What tells us only, if it is a member in the case of $x = 1$ or if it is not a member in $x = 0$, called the classical logic.

On the other hand, when $\mu_A(x)$ can associate values throughout the interval [0,1], this is called how fuzzy logic.

That said, two key concepts for our study can be denoted. The first is the fuzzy set, which implies the fuzziness of membership from x to A. And the second is the membership function, which tells us the level of belonging from x to A.

In relation to fuzzy sets, they propose to solve the problem of the ambiguity of various themes of human life. This define belongs to certain situations with the object or fact in question, such as linguistic values or values of membership.

There are several membership functions that allow modelling different behaviours; these can include the whole fuzzy set or part of it. This research is used the Gaussian membership function; it has two parameters. The first is μ which represents the center of the function. And δ is the width of the membership function Rutkowski (2004).

$$\mu_{A_n}(x) = e^{-\left(\frac{x-\mu}{\delta}\right)^2} \tag{1}$$

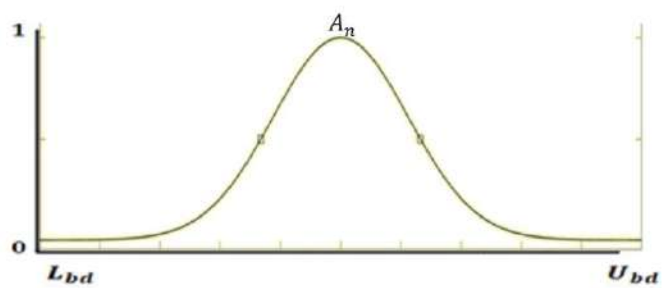

Figure 1. Gaussian Membership Function
Source: Own elaboration.

2.1 GARCH Model

The Generalized Autoregressive Conditional Heteroskedasticity GARCH (p, q) models are used to capture the behavior of the variance in the finance time series (Bollerslev, 1986). This method is categorized as an asymmetric model because assumed that the conditional variance is caused by the magnitude of the signal, and not for the negative and positive information. For example, if there is bad news the volatility is higher than other cases and can this methodology not take this information to make the forecast. The GARCH model is represented as:

$$y_t = \sigma(t)\varepsilon(t)$$
$$\sigma^2(t) = \omega + \sum_{i=1}^{p} \alpha_i \, y^2(t-i) + \sum_{j=1}^{q} \beta_j \, \sigma^2(t-j) \tag{2}$$

Where y_t is a stochastic time series process determined by the $\sigma(t)$ volatility function and $\varepsilon(t)$ a white noise. The conditional variance is represented thorough the equation $\sigma^2(t)$, and this is in the function of the square lags of y_t and its own lags. This method must comply with the next conditions:

$$\omega > 0$$
$$\alpha_i \geq 0$$
$$\beta_j \geq 0$$
$$\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1$$
$$i = 1, \cdots p; \;\; j = 1, \cdots, q.$$

The importance of this model is because can make a forecast of the various exchange rate, but also, for is possible combined this technique with fuzzy theory and make a different forecast of the traditional GARCH.

2.2 EGARCH model.

From Nelson (1991) the Exponential Generalized Autoregressive Conditional Heteroskedasticity EGARCH (p, q) is categorized as to how asymmetric model because this method recognized the magnitude and the sign of the information that causes the volatility of the time series. The importance of this methodology is that the conditional variance is the logarithm of the linear combination of y_t and its own lags.

This model is developed because sometimes the construction of the non-negativity of the parameters of the GARCH model is necessary, this when the estimation of the parameters falls in the restriction that the sum of them is greater than one. Therefore, under this methodology, the problem of restriction is solved. The variance of the model is determined for the following equations:

$$y_t = \sigma(t)\varepsilon(t)$$
$$\ln\left(\sigma^2(t)\right) = \omega + \sum\nolimits_{i=1}^{p} \alpha_i \, g(y_{t-i}) + \sum\nolimits_{j=1}^{q} \beta_j \, \sigma^2(t-j) \tag{3}$$

The section $\sum_{j=1}^{q} \beta_j \, \sigma^2(t-j)$ specifies the GARCH part of the methodology plus the function that allows modeling the condition of asymmetry in the variance, defined as $g(y_{t-i})$.

$$g(y_{t-i}) = \theta_1 y_{t-i} + \theta_2[|y_{t-i}| - E|y_{t-i}|] \tag{4}$$

Where the sign effect is $\theta_1 y_t$, in other words, the impact of the information whether positive or negative. Moreover, the magnitude is $\theta_2[|y_t| - E|y_t|]$ this equation denotes that both events of high or low volatility have influence in the forecast of the financial time series (Karlsson, 2002).

3. Model formulation of the Fuzzy GARCH with Gaussian Parameters

The important increase in volatility in the Foreign Exchange market causes that the forecast of these variables is increasingly complicated. In this way, the Fuzzy EGARCH with the Gaussian Parameters model is proposed to improve the predictive values of the exchange rates. For understanding this method, it is necessary to know the traditional GARCH (p, q) model (Bollerslev, 1986).

Assumption 1. The membership function of the GARCH parameters (ω, α_i and β_j) is a Gaussian type. In this way, it can be expressed as:

$$\mu_{A_1}(\omega_k) = e^{-\left(\frac{\omega_k - \omega}{\delta_\omega}\right)^2} \tag{5}$$

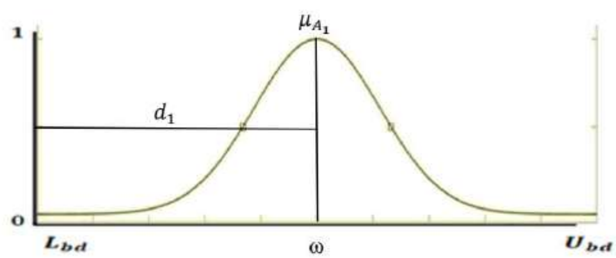

Figure 2. Gaussian Membership Function of ω_k

Source: Own elaboration.

$$d_1 = 3 * \delta_\omega - \omega$$

where $\mu_{A_1}(\omega_k)$ represents the membership function of the fuzzy set of the parameter ω_k, ω is the center and δ_ω is the variance. And d_1 is the distance where there is 99% of all parameters in this membership function. In this research is used the letter k how the subscript that defines an element of the membership function associated.

$$\mu_{A_{1+i}}(\alpha_{ik}) = e^{-\left(\frac{\alpha_{ik} - \alpha_i}{\delta_{\alpha_i}}\right)^2} \tag{6}$$

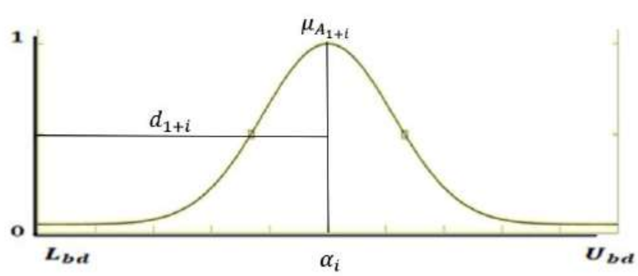

Figure 3. Gaussian Membership Function of α_i

Source: Own elaboration.

$$d_{1+i} = 3 * \delta_{\alpha_i} - \alpha_i$$

where $\mu_{A_{1+i}}(\alpha_{ik})$ represents the membership function of the fuzzy set of the parameter α_{ik}, α_i is the center and δ_{α_i} is the variance. And d_{1+i} is the distance where there is 99% of all parameters in this membership function.

$$\mu_{A_{1+i+j}}(\beta_{jk}) = e^{-\left(\frac{\beta_{jk}-\beta_j}{\delta_{\beta_j}}\right)^2} \tag{7}$$

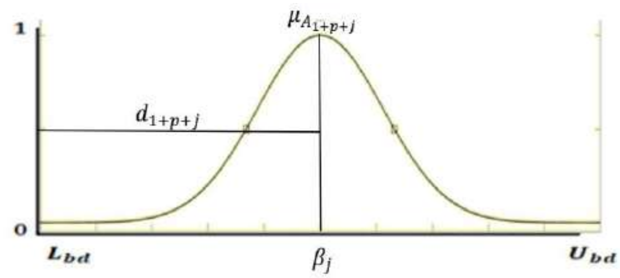

Figure 4. Gaussian Membership Function of β_j
Source: Own elaboration.

$$d_{1+p+j} = 3 * \delta_{\beta_j} - \beta_j$$

where $\mu_{A_{1+i+j}}(\beta_{jk})$ represents the membership function of the fuzzy set of the parameter β_{jk}, β_j is the center and δ_{β_j} is the variance. And d_{1+p+j} is the distance where there is 99% of all parameters in this membership function.

Assumption 2. If the $\sigma^2(t)$ conditional variance of a GARCH process is a fuzzy function. Then it can be expressed as:

$$\sigma^2(t) = \mu_{A_1}(\omega_k) + \sum_{i=1}^{p} \mu_{A_{1+i}}(\alpha_{ik})\, y^2(t-i) + \sum_{j=1}^{q} \mu_{A_{1+p+j}}(\beta_{jk})\, \sigma^2(t-j) \tag{8}$$

where (8) represents the fuzzy conditional variance of the GARCH process. In this case is necessary to define an error minimization function, for the present research is taken the Mean Absolut Error (MAD) and can be expressed as follows:

$$\epsilon = \frac{E[y_t - \hat{y}_t]}{n}$$

Then the problem is finding the fuzzy GARCH parameters, this can be obtained through solving the next linear programming problem:

$$\begin{array}{lll} \text{Minimize} & \epsilon & \\ & d_1 & > 0 \\ \text{Subject to} & d_{1+i} & > 0 \\ & d_{1+p+j} & > 0 \end{array} \quad (9)$$

When generating the parameters that guarantee the minimum ϵ also is found the non-fuzzy forecast of the conditional variance. And the Fuzzy GARCH model can be expressed as:

$$y_t = \sigma(t)\varepsilon(t)$$

$$\sigma^2(t) = \omega_k + \sum_{i=1}^{p} \alpha_{ik}\, y^2(t-i) + \sum_{j=1}^{q} \beta_{jk}\, \sigma^2(t-j) \quad (10)$$

This method must comply with the next conditions:

$$\begin{gathered} \omega_k > 0 \\ \alpha_{ik} \geq 0 \\ \beta_{jk} \geq 0 \\ \sum_{i=1}^{q} \alpha_{ik} + \sum_{j=1}^{p} \beta_{jk} < 1 \\ i = 1, \cdots p;\ \ j = 1, \cdots, q. \end{gathered}$$

The proposed method has the next phases:

I. Estimate the parameters of the GARCH with the Maximum Likelihood method.
II. Use the results of the previous step how the centre of the membership functions (5), (6) and (7).
III. Define the distance in the model d_n of each membership function and find the standard deviation as:

$$\delta_\varphi = \frac{d_n + \varphi}{3}$$

Where n is the lag associated with the parameter φ and δ_φ is the standard deviation of the membership function.

IV. Obtain the probability of the membership function with δ_φ, d_n and φ.
V. Determine with the Gaussian function the parameter φ_k using the values φ, δ_φ and the probability of the last step.
VI. Result of the linear programming problem (9).
VII. Finally, calculate the fuzzy Gaussian GARCH forecast.

4. Model formulation of the Fuzzy EGARCH with Gaussian Parameters

In the present section is developed the Fuzzy Exponential Generalized Autoregressive Conditional Heteroskedasticity Fuzzy EGARCH (p, q). This method can also be categorized how an asymmetric model because this method recognized the magnitude and the sign of the information that causes the volatility in financial time series. From the model of Nelson (1991), it is assumed that their parameters have a Gaussian membership function.

Assumption 3. The membership function of the EGARCH parameters $(\omega, \alpha_i, \beta_j, \theta_1$ and $\theta_2)$ is a Gaussian type. In this way, it can be expressed as (5), (6), (7) and:

$$\mu_{B_1}(\theta_{1k}) = e^{-\left(\frac{\theta_{1k}-\theta_1}{\delta_{\theta_1}}\right)^2} \quad (11)$$

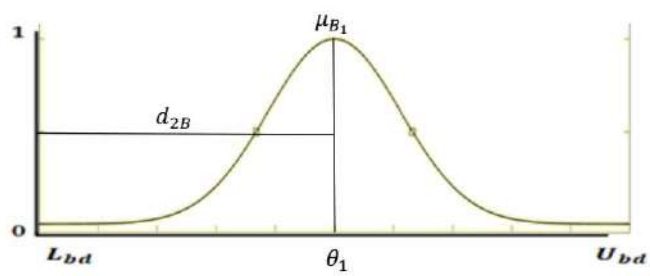

Figure 5. Gaussian Membership Function of θ_1
Source: Own elaboration.

$$d_{1B} = 3 * \delta_{\theta_1} - \theta_1$$

where $\mu_{B_1}(\theta_{1k})$ represents the membership function of the fuzzy set of the parameter θ_{1k}, θ_1 is the center and δ_{θ_1} is the variance. And d_{1B} is the distance where exist the 99% of all parameters in this membership function.

$$\mu_{B_2}(\theta_{2k}) = e^{-\left(\frac{\theta_{2k}-\theta_2}{\delta_{\theta_2}}\right)^2} \quad (12)$$

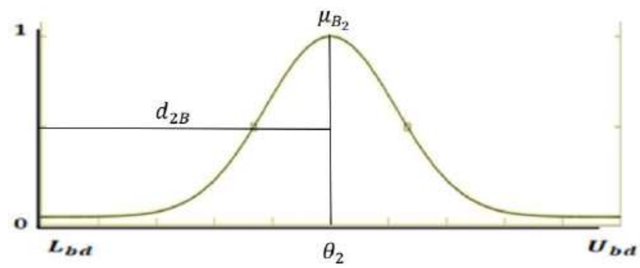

Figure 6. Gaussian Membership Function of θ_2
Source: Own elaboration.

$$d_{2B} = 3 * \delta_{\theta_2} - \theta_2$$

where $\mu_{B_2}(\theta_{2k})$ represents the membership function of the fuzzy set of the parameter θ_{2k}, θ_2 is the center and δ_{θ_2} is the variance. And d_{2B} is the distance where exist the 99% of all parameters in this membership function.

Assumption 4. If the $\sigma^2(t)$, the conditional variance of an EGARCH process is a fuzzy function. Then it can express as:

$$\sigma^2(t) = \mu_{A_1}(\omega_k) + \sum_{i=1}^{p} \mu_{A_{1+i}}(\alpha_{ik})\, y^2(t-i) + \sum_{j=1}^{q} \mu_{A_{1+p+j}}(\beta_{jk})\, \sigma^2(t-j)$$
$$g(y_{t-i}) = \mu_{B_1}(\theta_{1k}) y_{t-i} + \mu_{B_2}(\theta_{2k})[|y_{t-i}| - E|y_{t-i}|] \quad (13)$$

where (13) represents the fuzzy conditional variance of the EGARCH process. In this case is necessary to define an error minimization function, for the present research is taken the Mean Absolut Error (MAD) and can be expressed as follows:

$$\epsilon = \frac{E[y_t - \hat{y}_t]}{n}$$

Then the problem is finding the fuzzy EGARCH parameters that can obtain resolving the next linear programming problem:

$$\begin{array}{lll} \text{Minimize} & \epsilon & \\ & d_1 & > 0 \\ & d_{1+i} & > 0 \\ \text{Subject to} & d_{1+p+j} & > 0 \\ & d_{1B} & > 0 \\ & d_{2B} & > 0 \end{array} \quad (14)$$

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When are establish the parameters that guarantee the minimum ϵ, the non-fuzzy forecast of the conditional variance is found. Then the Fuzzy EGARCH model can be expressed as follows:

$$y_t = \sigma(t)\varepsilon(t)$$

$$\ln\big(\sigma^2(t)\big) = \omega_k + \sum_{i=1}^{p} \alpha_{ik}\, g(y_{t-i}) + \sum_{j=1}^{q} \beta_{jk}\, \sigma^2(t-j) \qquad (15)$$

$$g(y_{t-i}) = \theta_{1k} y_{t-i} + \theta_{2k}[|y_{t-i}| - E|y_{t-i}|]$$

This technic must comply with the following conditions:

$$\omega_k > 0$$
$$\alpha_{ik} \geq 0$$
$$\beta_{jk} \geq 0$$
$$\theta_{1k} \geq 0$$
$$\theta_{2k} \geq 0$$
$$i = 1, \cdots p; \;\; j = 1, \cdots, q.$$

The proposed model has the next phases:

I. Estimate the parameters of the EGARCH with the Maximum Likelihood method.
II. Use the results of the previous step how the centre of the membership functions (5), (6), (7), (11) and (12).
III. Define the distance in the model d_n of each membership function and find the standard deviation as:

$$\delta_\varphi = \frac{d_n + \varphi}{3}$$

Where n is the lag associated with parameter φ and δ_φ is the standard deviation of the membership function.

IV. Obtain the probability of the membership function with δ_φ, d_n and φ.
V. Determine with the Gaussian function the parameter φ_k using the values φ, δ_φ and the probability of the last step.
VI. Result of the linear programming problem (14).
VII. Finally, calculate the fuzzy Gaussian EGARCH forecast.

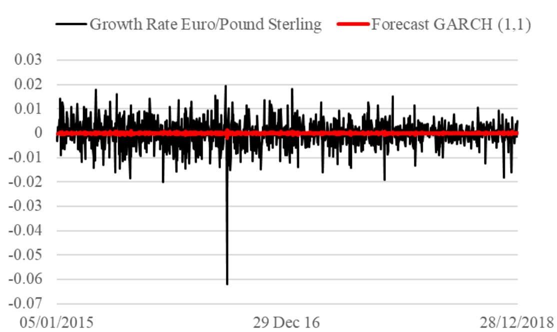
5. Application to forecast foreign exchange market

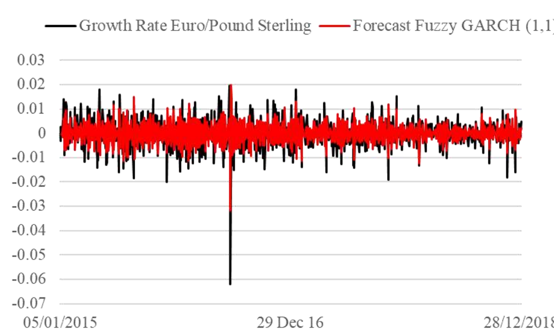
In this section has applied the model to forecasting the foreign exchange rate of MX peso against US dollar (2035 observations), Euro against Pound Sterling (964 observations), Frank Switzerland against US dollar (2035 observations) and Yen against Euro (2034 observations). This work used the growth rate of the time series data in a daily format from January 2015 to November 2022. And for making them out sample test is from January 2019 and December 2022 (42 observations) from Bank of England and Banco de México.

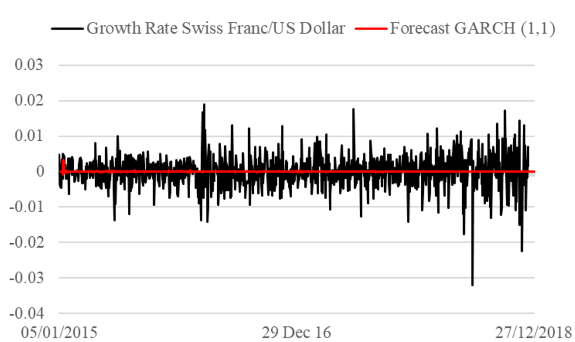
First are estimated the parameters of the GARCH (1, 1) and EGARCH (1, 1) with the Maximum Likelihood method for the four exchange rates. After is used the results of the previous step how the centre of the Gaussian Membership Functions of the parameters of each model. Define the distance of each model in the membership function and find the standard deviation. Resolve the linear programming problem (9) and (14) for both models and make the forecast.

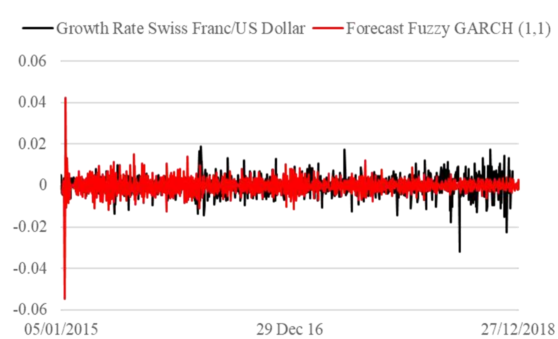
5.1 Forecast Fuzzy GARCH

The forecast of the GARCH (1,1) for the Growth rate of the exchange rates MX peso versus US dollar, Euro versus Pound Sterling, Frank Switzerland versus US dollar and Yen versus Euro are developed in this section together with the comparison of the proposed Gaussian parameters fuzzy GARCH model and the traditional GARCH.









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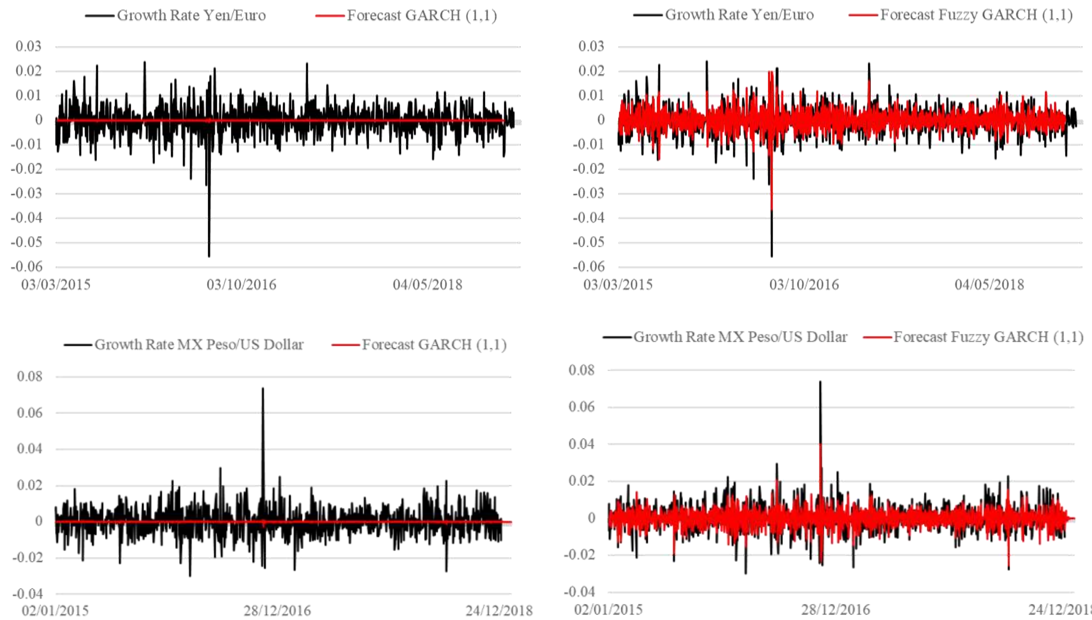


Figure 7. Forecast comparison in the sample of the GARCH (1, 1) and fuzzy GARCH (1, 1) models (sample 2015-2018)

Source: Own elaboration in Excel.

First, Figure 7 shows the different forecasts developed in this research, on the left side are presented the forecast in the sample of the GARCH (1,1) that are the white lines, and the growth rate of the exchange rates are the black ones. These figures present that the forecast is not accurate to the real variable. On the other hand, the prediction in the sample of the Fuzzy GARCH (1, 1) is the white lines and the black ones are the growth rate of the exchange rates. This Figure denoted that the fuzzy forecasts have better accuracy in comparison with the GARCH (1, 1) model.

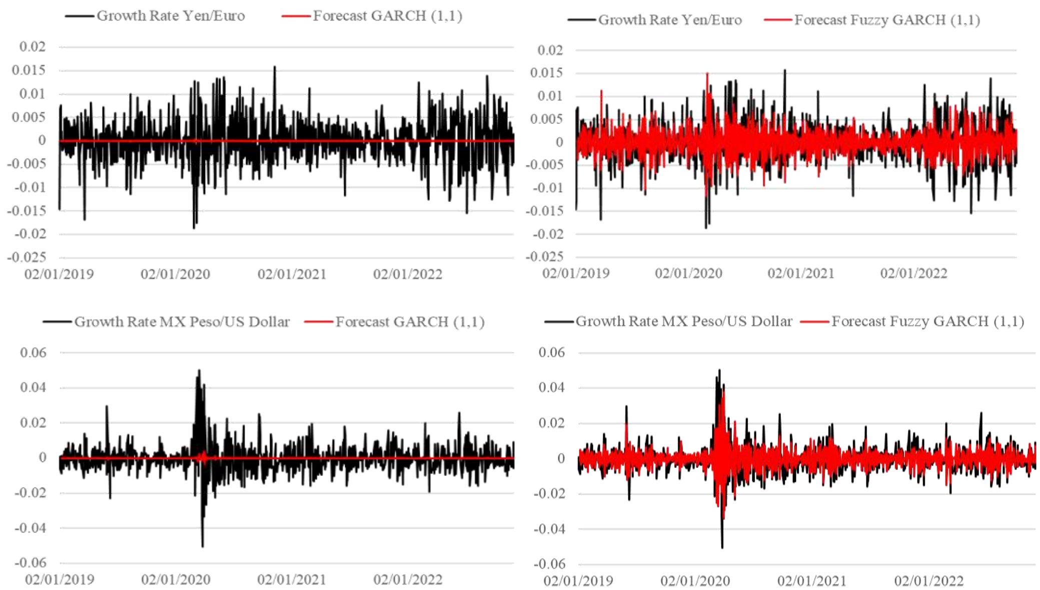


Figure 8 (continuation). Forecast comparison in the sample of the GARCH (1, 1) and fuzzy GARCH (1, 1) models (sample 2019-2022)
Source: Own elaboration in Excel.

Furthermore, the table 1 shows the performance comparison of the mean absolute deviation for the growth rate of Mexican Peso/ American Dollar in the sample is 0.57% of daily error for the GARCH (1,1) and 0.42% for the Fuzzy GARCH (1,1) there is a difference of 0.15% of lower error in the proposed model. In the case of the Yen/Euro, the GARCH (1, 1) have 0.11 more error than the Fuzzy GARCH (1, 1) model. The mean absolute deviation of the Swiss Franc/ US Dollar is of 0.41% for the GARCH (1,1) and 0.30% in the Fuzzy GARCH (1,1) this represents that the suggested method has 0.11% lower deviation than the traditional model and finally the Euro/ Pound Sterling presented 0.41% for the GARCH (1,1) and 0.30% in the Fuzzy GARCH (1,1). On average the model-based in fuzzy theory presents a 0.11% lower error than its similar basic model.

For the out sample test in the table 1 shows that mean absolute deviation for the growth rate of Mexican Peso/ American Dollar in the sample is 0.34% of daily error for the GARCH(1,1) and 0.37% for the Fuzzy GARCH (1,1), in the case of the Yen/Euro is 0.35% for the GARCH(1,1) and 0.35% in the Fuzzy GARCH (1,1), the Swiss Franc/ US Dollar is 0.28% for the GARCH(1,1) and 0.26% in the Fuzzy GARCH (1,1) and finally the Euro/ Pound Sterling presented an error daily of 0.34% for the GARCH(1,1) and 0.37% in the Fuzzy GARCH (1,1). These results and Figure 8 denoted that the Fuzzy GARCH model has more variability in the forecast out the sample and this cause that the mean absolute deviation in some cases is lower in the GARCH (1,1) than the proposed model.

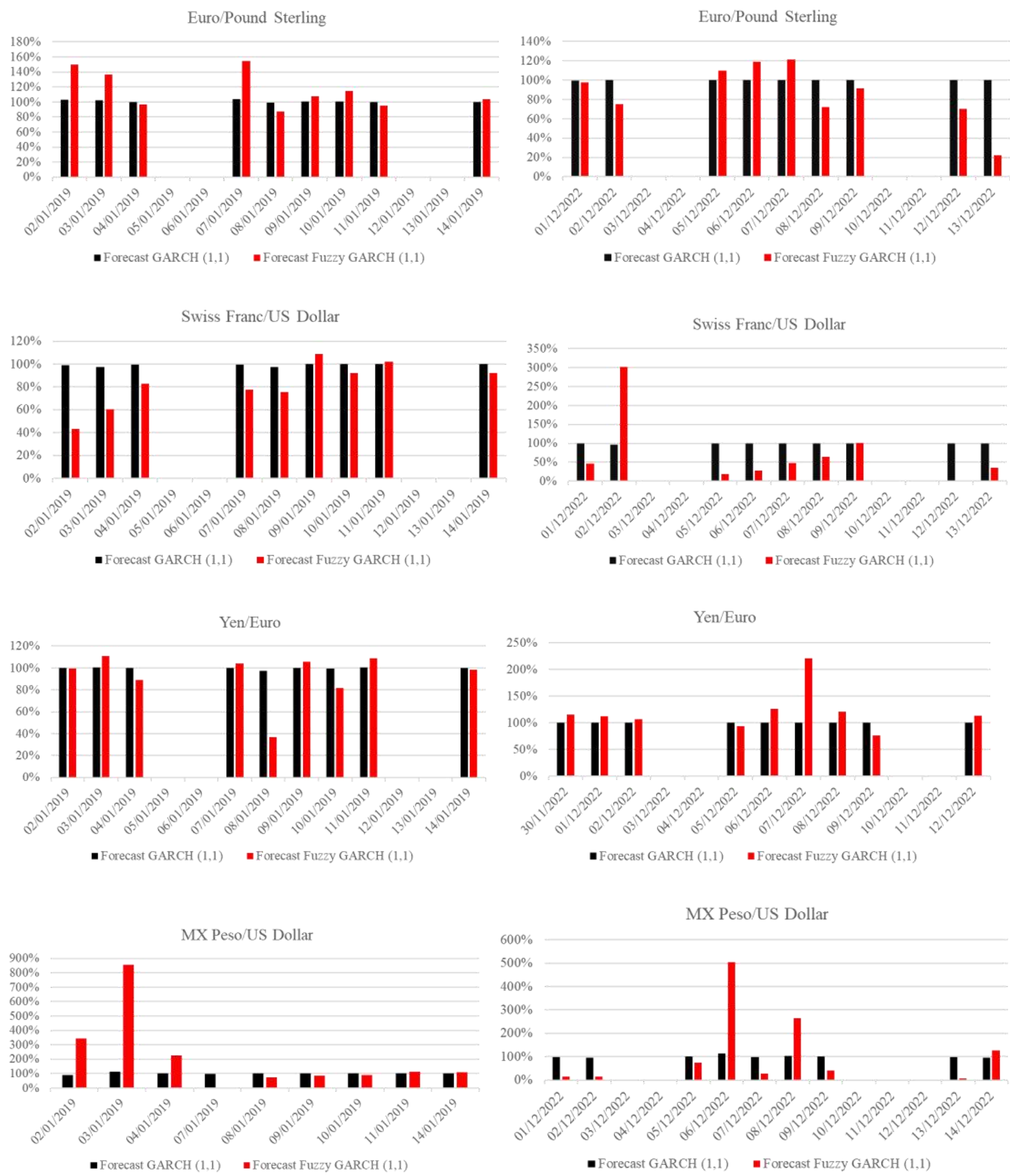


Figure 9. Percentage error comparison out the sample of the GARCH (1, 1) and fuzzy GARCH (1, 1) models
Source: Own elaboration in Excel.

These results said that the suggested model Fuzzy GARCH (1, 1) gives a good estimate of the exchange rates than the GARCH (1, 1) in the sample. And out of the sample, the fuzzy model let's make a forecast with higher variability that the benchmark model.

Table 1. Mean Absolute Deviation comparison of the GARCH (1, 1) and Fuzzy GARCH (1, 1) Models

Model		**In sample test**		**Out sample test**	
Sample		(2015-2018)	(2019-2022)	(Jan-2019)	(Dic-2022)
GARCH (1,1)	**¥\€**	0.4396	0.3367	0.3553	0.5049
FUZZY GARCH (1,1)	**¥\€**	0.3255	0.2577	0.3572	0.5162
GARCH (1,1)	**€\£**	0.4166	0.3458	0.4225	0.3460
FUZZY GARCH (1,1)	**€\£**	0.3067	0.2443	0.4393	0.3326
GARCH (1,1)	**SFr\US**	0.4164	0.7531	0.2858	0.3437
FUZZY GARCH (1,1)	**SFr\US**	0.3066	0.2500	0.2645	0.1014
GARCH (1,1)	**MX\US**	0.5732	0.5365	0.3493	0.5428
FUZZY GARCH (1,1)	**MX\US**	0.4286	0.4060	0.3774	0.4052

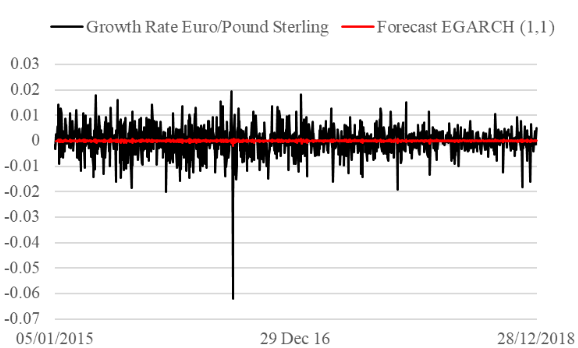
Source: Own elaboration in Excel.

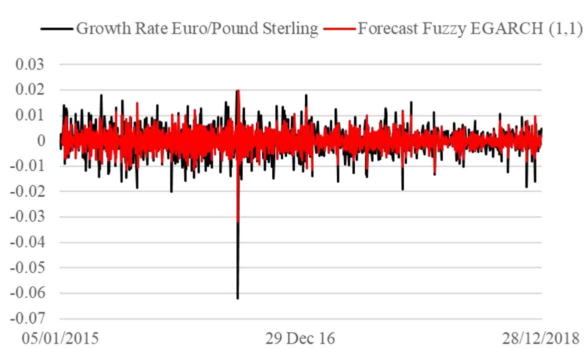
5.2 Forecast Fuzzy EGARCH

Another fuzzy structure is the Gaussian parameters fuzzy EGARCH model; this is applied to the exchange rates, MX peso versus US dollar, Euro versus Pound Sterling, Frank Switzerland versus US dollar and Yen versus Euro.

Figure 9 shows the forecasts in the sample of the EGARCH (1, 1) on the left side, these are the white lines, and the growth rates of the exchange rates are the black ones. These figures illustrated a bad forecast in comparison with the values of the exchange rates. On the left side is presented the estimate in the sample of the Fuzzy EGARCH (1, 1) which are the white lines, and the black are the growth rate of the exchange rates. The Fuzzy EGARCH (1,1) give a better forecast than the GARCH (1,1) model.

Additionally, the table 2 shows the performance comparison of the mean absolute deviation for the growth rate of Mexican Peso/ American Dollar in the sample is 0.57% of daily error for the GARCH (1,1) and 0.42% for the Fuzzy GARCH (1,1) there is a difference of 0.15% of lower error in the proposed model. In the case of the Yen/Euro, the GARCH (1, 1) have 0.12% more error than the Fuzzy GARCH (1, 1) model. The mean absolute deviation of the Swiss Franc/ US Dollar is of 0.41% for the GARCH (1,1) and 0.30% in the Fuzzy GARCH (1,1) this represents that the suggested method has 0.11% lower deviation than the traditional model and finally the Euro/ Pound Sterling presented 0.41% for the GARCH (1,1) and 0.30% in the Fuzzy GARCH (1,1). On average the model-based in fuzzy theory presents a 0.11% lower error than its similar basic model.





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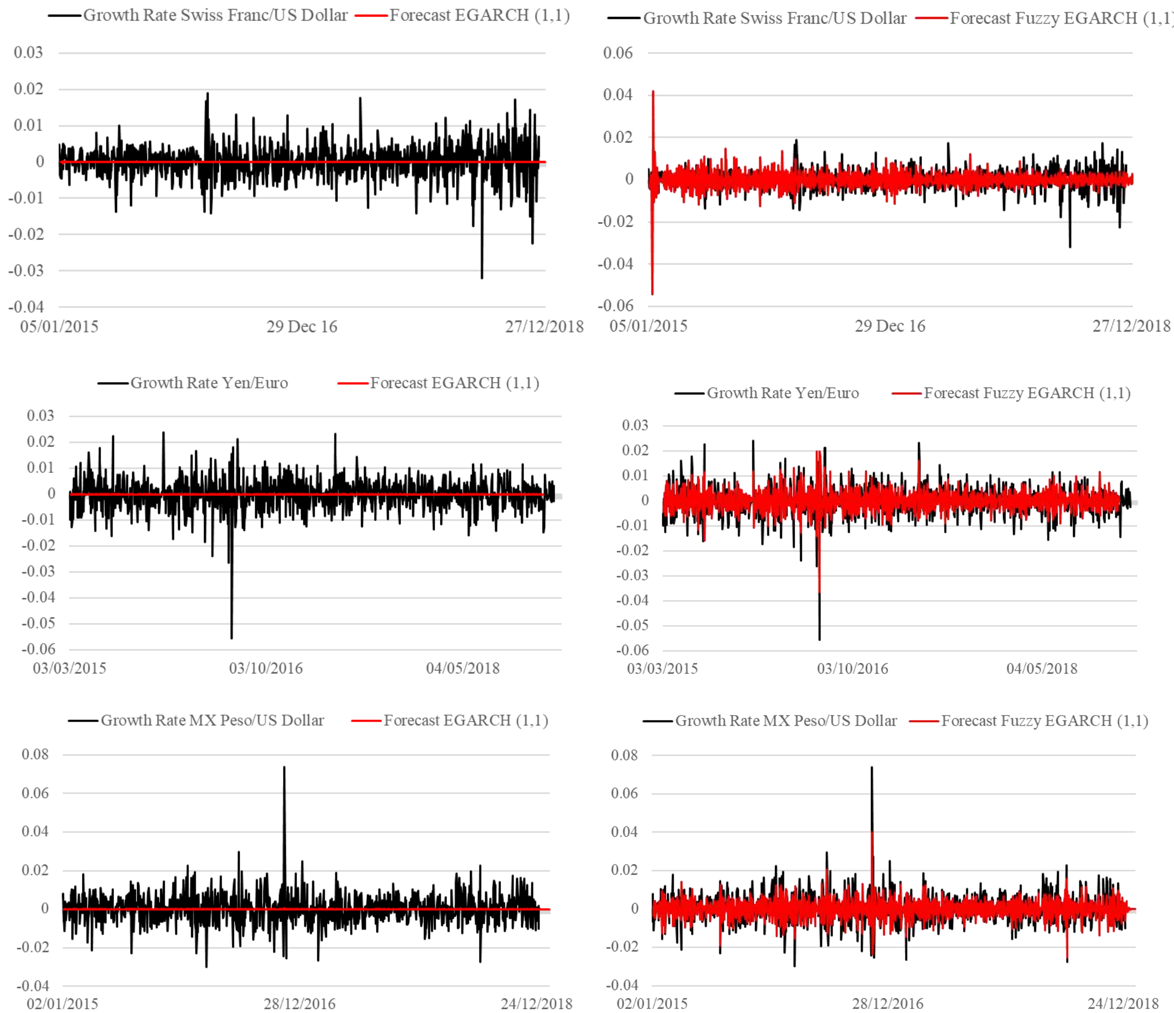
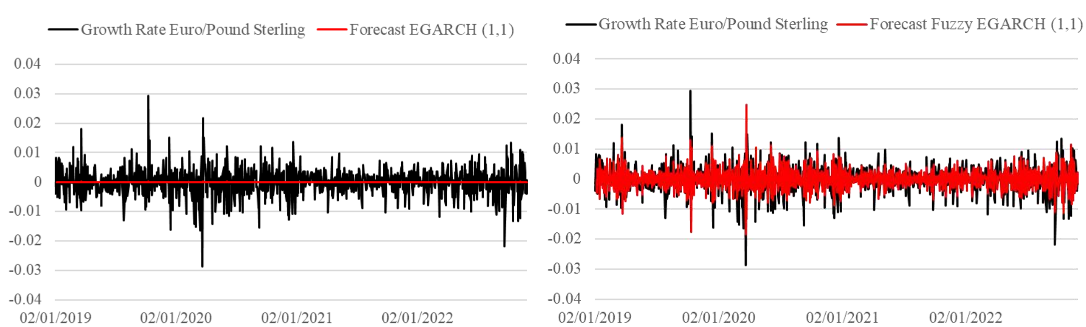


Figure 10. Forecast comparison in the sample of the EGARCH (1, 1) and fuzzy EGARCH (1, 1) Models (sample 2015-2018)
Source*:* Own elaboration in Excel.



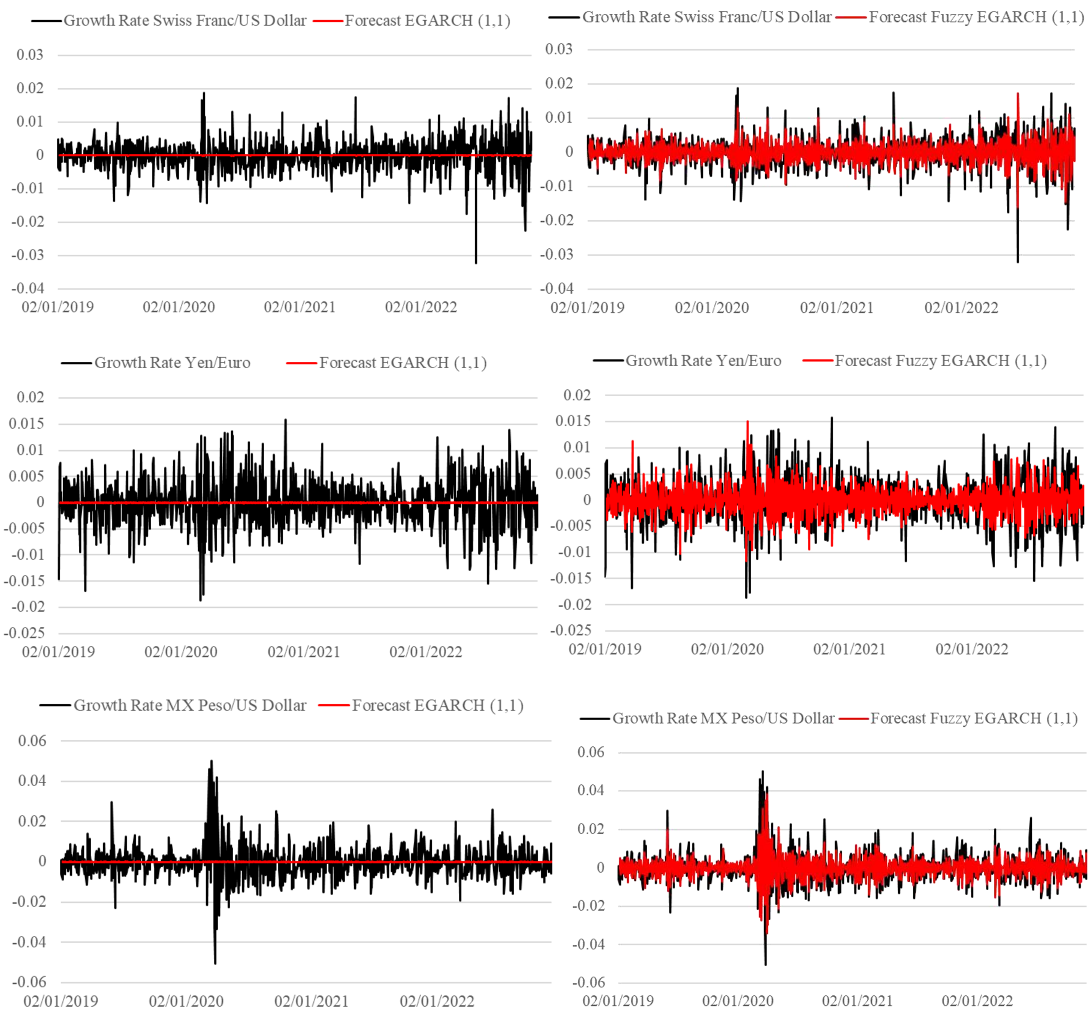


Figure 11 (continuation). Forecast comparison in the sample of the EGARCH (1, 1) and fuzzy EGARCH (1, 1) Models (sample 2019-2022)
Source*:* Own elaboration in Excel.

For the out sample test in the table 2 shows that mean absolute deviation for the growth rate of Mexican Peso/ American Dollar in the sample is 0.34% for the EGARCH(1,1) and 0.37% to the Fuzzy EGARCH (1,1), in the case of the Yen/Euro is 0.35% for the EGARCH(1,1) and 0.35% in the Fuzzy EGARCH (1,1), the Swiss Franc/ US Dollar is 0.28% for the EGARCH(1,1) and 0.26% in the Fuzzy EGARCH (1,1) and finally the Euro/ Pound Sterling presented an error daily of 0.34% for the EGARCH(1,1) and 0.37% in the Fuzzy EGARCH (1,1). These results and the Figure 9 denoted that the Fuzzy GARCH model has more variability in the forecast out the sample and this cause that the mean absolute deviation in some cases is lower in the EGARCH (1,1) than the proposed model.

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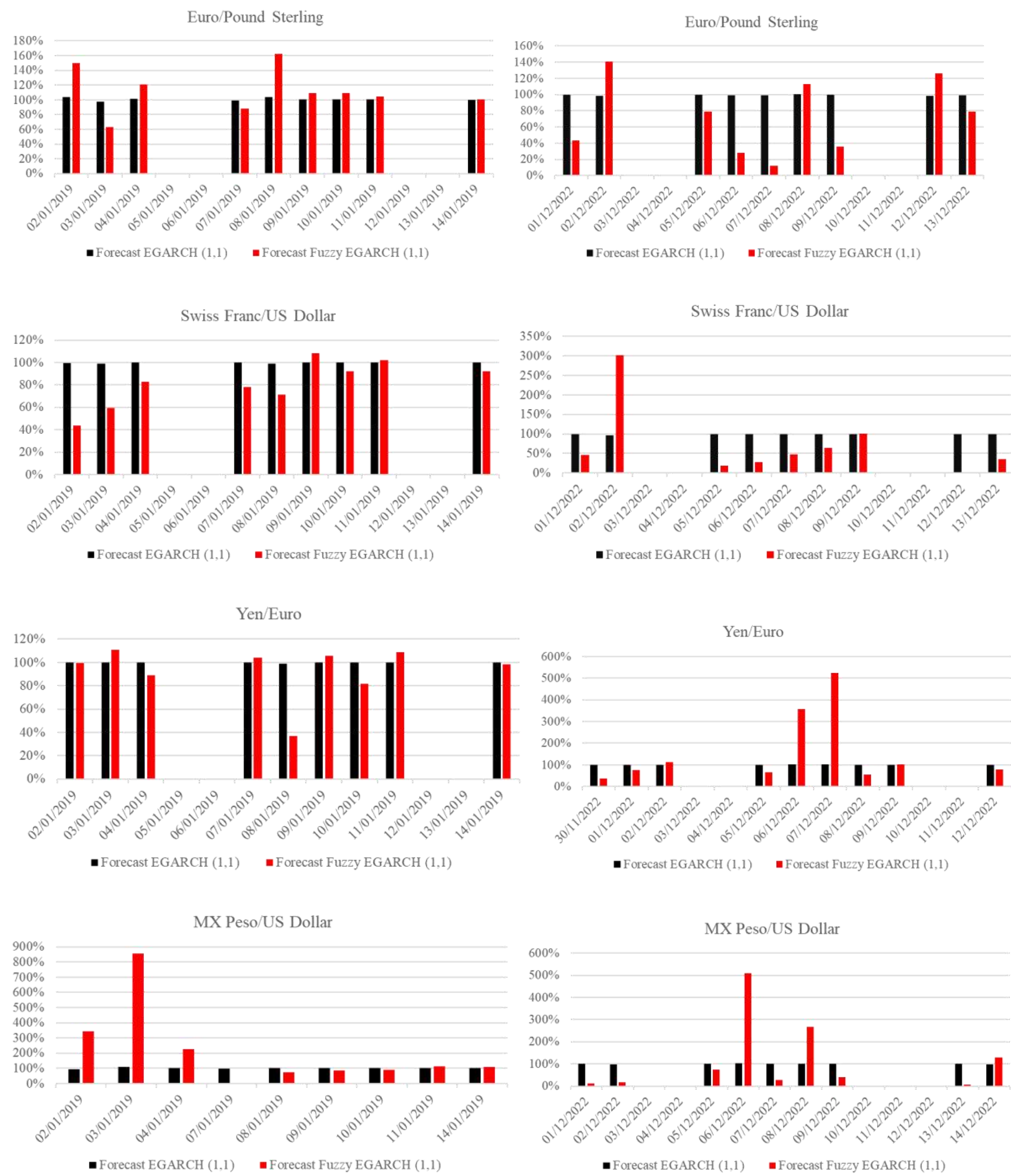

Figure 12. Percentage error comparison out the sample of the EGARCH (1, 1) and fuzzy EGARCH (1, 1) models

Source: Own elaboration in Excel.

These results said that the suggested model Fuzzy EGARCH (1, 1) estimates the exchange rates better than the EGARCH (1, 1) in the sample. And out-sample the fuzzy model let's make a forecast with major variability that the benchmark model.

Table 2. Mean Absolute Deviation comparison of the EGARCH (1, 1) and Fuzzy EGARCH (1, 1).

Model		**In sample test**		**Out sample test**	
Sample		(2015-2018)	(2019-2022)	(Jan-2019)	(Dic-2022)
EGARCH (1,1)	**¥\€**	0.4420	0.3509	0.3553	0.5039
FUZZY EGARCH (1,1)	**¥\€**	0.3255	0.2677	0.3572	0.4174
EGARCH (1,1)	**€\£**	0.4167	0.3527	0.4213	0.3453
FUZZY EGARCH (1,1)	**€\£**	0.3068	0.2413	0.4392	0.3134
EGARCH (1,1)	**SFr\US**	0.4101	0.3312	0.2860	0.3476
FUZZY EGARCH (1,1)	**SFr\US**	0.3015	0.2467	0.2646	0.2703
EGARCH (1,1)	**MX/US**	0.5748	0.5396	0.3494	0.5389
FUZZY EGARCH (1,1)	**MX/US**	0.4287	0.4059	0.3774	0.3761

Source: Own elaboration in Excel.

Finally, this research found that the hybrid models Fuzzy GARCH and Fuzzy EGARCH estimate better the behaviour of the exchange rates that the traditional model GARCH and EGARCH. In this situation, the proposed models have the capacity to generate a good forecast for high volatility variables and financial time series.

6. Conclusions

In this research, based on the Conditional Heteroskedasticity models, it was suggested new techniques (Fuzzy GARCH and Fuzzy EGARCH) and apply it to estimate the foreign exchange market in four cases of MX pesos against EE. UU Dollar, Euro against Pound Sterling, Swiss Franc against US Dollar, and Yen against the Euro. The Fuzzy GARCH and Fuzzy EGARCH models show major effectiveness for forecast the behaviour of the volatility in the exchange rates that the simple method GARCH and EGARCH.

Though the concepts of the GARCH and EGARCH are used to designed and formulate the Fuzzy GARCH and Fuzzy EGARCH models, the parameters of the proposed methods are fuzziness to release the assumption of Fuzzy Conditional Variances. But in the proposed models the output is non-fuzzy because the linear optimization formulated in the method allows finding the parameters that generate the best forecast of the input.

This research found that the models based on fuzzy theory have better estimates of volatility in financial time series. This allows developing new prediction methods based on the structure of fuzzy logic, it is also necessary to establish an analysis of a greater number of data to try to discriminate if the effect of the error effect decreases.

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Annex

Table A3. Sum of square error comparison of the GARCH (1, 1) and Fuzzy GARCH (1, 1) Models

Model	**In sample test**	
Sample	(2015-2018)	(2019-2022)
GARCH (1,1) ¥\€	0.0199	0.0347
FUZZY GARCH (1,1) ¥\€	0.0117	0.0182
GARCH (1,1) €\£	0.0219	0.0331
FUZZY GARCH (1,1) €\£	0.0108	0.0179
GARCH (1,1) SFr\US	0.1017	0.0453
FUZZY GARCH (1,1) SFr\US	0.0113	0.0262
GARCH (1,1) MX/US	0.0604	0.0602
FUZZY GARCH (1,1) MX/US	0.0332	0.0339

Source: Own elaboration in Excel.

Table A2. Sum of square error comparison of the GARCH (1, 1) and Fuzzy GARCH (1, 1) Models

Model	**In sample test**	
Sample	(2015-2018)	(2019-2022)
EGARCH (1,1) ¥\€	0.0200	0.0351
FUZZY EGARCH (1,1) ¥\€	0.0117	0.0182
EGARCH (1,1) €\£	0.0233	0.0331
FUZZY EGARCH (1,1) €\£	0.0108	0.0179
EGARCH (1,1) SFr\US	0.0209	0.0453
FUZZY EGARCH (1,1) SFr\US	0.0113	0.0261
EGARCH (1,1) MX/US	0.0617	0.0605
FUZZY EGARCH (1,1) MX/US	0.0332	0.0339

Source: Own elaboration in Excel.