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An application of the Black-Scholes-Merton (Osborne-Samuelson) Model to the Mexican Stock Exchange

Castañeda-Leyva, Netzahualcóyotl
Sáinz-Fernández, Manuel Delfino
Trejo-Pech, Carlos Omar

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Abstract

This paper contributes to a better understanding of the Mexican Stock Exchange (MSE) through an empirical application of the Black-Scholes-Merton and Vasicek models, complementing the limited literature related to modeling prices and portfolios of Mexican stocks. The models are used to estimate return and volatility parameters of stocks and to optimize portfolios maximizing the power utility function. Inference of parameters is based on maximum likelihood estimation (MLE). Overall, our results show that risk and returns parameters are consistent with the risk-return theoretical framework. Results of optimal portfolios are also consistent with a hypothetical investor's rationale and risk profile. All estimated optimal portfolios outperform the Mexican Stock Exchange index (IPC) during the 2006-2010 period of study.

Keywords:

Mexican Stock Exchange, Time series, Stochastic models, Black-Scholes-Merton, Vasicek, Optimal portfolios.

JEL classification:

C58, G17

Castañeda-Leyva, N. ✉ *Professor of Statistics, Department of Economics, Universidad de Guanajuato, México.* ✆ *+52 473 7352908. E-mail: netzacl70@gmail.com*

Sáinz-Fernández, M.D., Professor of Finance, School of Business and Economics, Universidad Panamericana at Aguascalientes, México. ✆ *+52 449 9106212. E-mail: msainz@up.edu.mx*

Trejo-Pech, C.O., Professor of Finance, School of Business and Economics, Universidad Panamericana at Guadalajara, México; Visiting Scholar at Purdue University, USA. ✆ *+52 331 3682218. E-mail: ctrejo@up.edu.mx*

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Una Aplicación del Modelo Black-Scholes-Merton (Osborne-Samuelson) **al Mercado de Valores Mexicano**

Castañeda-Leyva, Netzahualcóyotl
Sáinz-Fernández, Manuel Delfino
Trejo-Pech, Carlos Omar

Resumen

Este artículo contribuye a un mejor conocimiento del Mercado de Valores Mexicano mediante una aplicación empírica de los modelos Black-Scholes-Merton y Vasicek, complementando, de esta manera, la escasa literatura relacionada con la modelización de precios y carteras de acciones mexicanas. Los modelos anteriormente mencionados son utilizados tanto para estimar los parámetros relativos al rendimiento y la volatilidad de los valores como para la optimización de carteras, maximizando la función de utilidad tipo potencia. La inferencia sobre los parámetros está basada en la estimación máximo verosímil. En general, los resultados obtenidos muestran que los parámetros de riesgo y rendimiento son consistentes con el marco teórico. Los resultados de las carteras óptimas también son consistentes con el razonamiento de un hipotético inversor y su perfil de riesgo. Todas las carteras óptimas estimadas superan el Índice Bursátil Mexicano (IPC) durante el periodo objeto de este estudio: 2006-2010.

Palabras clave:

Mercado de Valores mexicano, serie temporal, Black-Scholes-Merton, modelos estocásticos, Vasicek, carteras óptimas.

1. Introduction

In this study the stock market (i.e., stock prices) is modeled as a multivariate geometric Brownian motion with constant parameters. The model is a generalization of the continuous time stochastic model known as the Black-Scholes-Merton model (Black and Scholes, 1973; Merton, 1971).[1] The bond market (i.e., Treasury bill prices), is in turn modeled with the continuous time stochastic model of interest rate by Vasicek (1977).[2] Parameter estimates, under maximum likelihood estimation (MLE), are used to optimize diversified portfolios under the widely known mean – variance Markowitz's framework. Portfolios are optimized using a power utility function for various investors' risk profiles under stochastic control. In particular, we study Mexican Stock Exchange and Mexican Treasury bill data over a seven year period.

This article contributes to the finance literature in two regards. First, in general, we consider this study to be an important empirical contribution with which to better understand this emerging market, as no other studies have attempted to model the Mexican Stock Exchange market using BSM.[3] In particular, some of the methods for the empirical implementation of this work are contributions to the literature. For instance, we provide a method for parameter estimation, with MLE, for time series data with asymmetrical incomplete data.[4] Another example of the empirical contribution is the dynamic weighted estimator (more in section 3.2) implemented for robustness testing. Second, this study could be of interest for practitioners as our findings indicate that our selected optimal portfolios had hypothetically outperformed, in a consistent way, the market index in terms of risk-return tradeoff. This could be of importance for conservative investors who are usually advised by portfolio managers to invest in tracking index stock funds.

The rest of the paper is organized as follows. Section 2 presents the models and describes the wealth process and portfolio optimization under stochastic control. Section 3 describes the databases and provides details for the estimation of parameters. Results are analyzed in section 4 and conclusions are provided in section 5.

[1] *The Black-Scholes-Merton (BSM) model is also known as the Osborne-Samuelson model (Osborne, 1959; Samuelson, 1965).*

[2] *The Vasicek model is also known as the Ornstein and Uhlenbeck model (Ornstein and Uhlenbeck, 1930).*

[3] *In a more general sense (no exclusive for studies on the Mexican Stock Exchange), in the finance literature most studies model stock prices with discrete rather than continuous time models.*

[4] *To the extent that researchers have missing observations in their dataset (a very common problem) this method could be of interest to other researchers.*

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2. Models

The models, both for equities and interest rates, are presented in this section. The portfolio optimization process under stochastic control is also described. While the solutions of these models were elaborated in Castañeda-Leyva *et al*. (2008), we put the models together to empirically test the portfolios of Mexican firms.

2.1. Black-Scholes-Merton model for equities

Consider a financial market composed by m risky assets and a risk free bond. The dynamics of prices for risky assets follow an m-dimensional Brownian Motion (B.M.) (Bachelier, 1900), also known as a Wiener (1958) process. Let S_t^i be the price of risky asset i at time t, $t \in [0,T]$, $i=1, \dots, m$. The price model (1) is a generalization of the BSM model (Black and Scholes, 1973, and Merton, 1971), according to the following stochastic differential equation (SDE):

$$dS_t^i = S_t^i(\mu_i\, dt + \sigma_{i1} dW_t^1 + \cdots + \sigma_{im} dW_t^m), \quad (1)$$

for $i=1, \dots, m,$ where

$$\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}, \; \sigma\sigma' = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1m} \\ \vdots & \ddots & \vdots \\ \sigma_{m1} & \cdots & \sigma_{mm} \end{pmatrix}, \; W_t = \begin{pmatrix} W_t^1 \\ \vdots \\ W_t^m \end{pmatrix}; \; t \geq 0,$$

are the vector of stock returns, the volatility or uncertainty matrix and the underlying m-dimensional B.M. or stochastic vector, respectively. It is assumed that the parameters of the model remain constant over time.

The solution of the SDE (1) (Castañeda-Leyva *et al*., 2008) using Itô's lemma is

$$S_t^i = S_0^i\, e^{\left(\mu_i - \frac{1}{2}\sigma_i^2\right) t + \sigma_i W_t^i}; \; t \geq 0, \quad (2)$$

with

$$\sigma_i = \sigma_{i1} + \cdots + \sigma_{im}$$

and

$$W_t^i = \frac{1}{\sigma_i}\,(\sigma_{i1} W_t^i + \cdots + \sigma_{im} W_t^i).$$

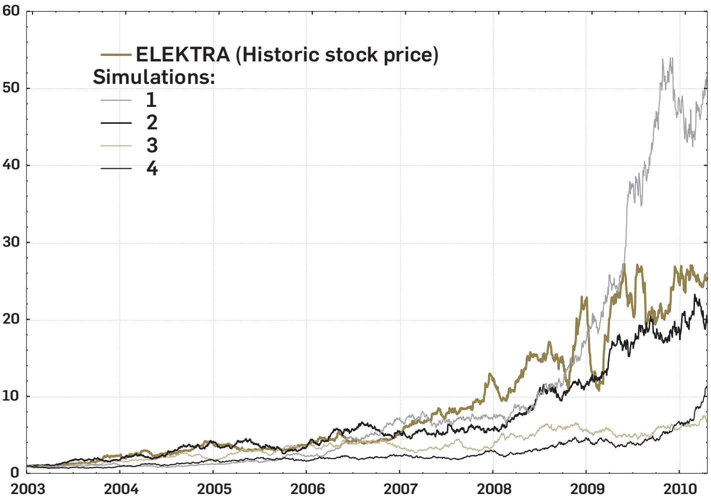
From (2) the price model S_t^i is always positive and it follows a log-Normal distribution. Moreover, the log-returns are Normal distributed:

$$\log \frac{S_{t+h_j}^i}{S_t^i} \sim N\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)h_j,\ \sigma_i^2 h_j\right), \text{ for all } h_j,\ t > 0 \text{ with } h_j = t_j - t_{j-1}.$$

This fact is given by $\{W_t^i\}$ being a unidimensional B.M. In consequence, this process has stationary and independent increments, and W_t^i is Normal distributed with mean zero and variance t, for all $t>0$.

A practical advantage of the model (1) is the simple way it simulates prices for the risky assets. For example, in Figure 1 the randomness of the stock price is shown with four simulated paths for ELEKTRA's stock prices, using the historical parameters (estimated as shown below in equation (3)).

Figure 1. **ELEKTRA's relative stock prices and four simulated paths using the BSM model solution of equation (2) with $S_0=1$**



SOURCE: DAILY DATA OF CLOSING PRICE OF ELEKTRA STOCK TAKEN FROM INFOSEL DATABASE FROM 01/02/2003 TO 04/23/2010 TO ESTIMATE STOCK PARAMETERS, $\hat{\mu}=50.3\%$ AND $\hat{\sigma}=35.2\%$ OF EQUATION (3)

2.1.1. Estimation of parameters of the BSM model

Model (1) has $m+\frac{m(m+1)}{2}$ parameters, with m corresponding to the vector μ and $\frac{m(m+1)}{2}$ to the symmetric matrix $\sigma\sigma'$. MLE is used to estimate the parameters μ and σ.

Define $h_j=t_j-t_{j-1}$ *; for* $j=1, \dots, n$. In particular, without missing data, $h_j=h=\frac{T}{n}=1$ day. Note that $\Sigma_{j=1}^n h_j=\Sigma_{j=1}^n (t_j-t_{j-1})=T$.

Assume that the price processes $\{S_t^i\}_{t\geq 0}$; for $i=1, \dots, m$, are observed in $n+1$ equidistant times:

$$0=t_0<t_1<\cdots<t_n=T, \text{ with } t_j=h_j,\ \ j=0, \dots, n,\ h=\frac{T}{n}.$$

Here h represents the time length of one day (i.e., each day from Friday to Monday, and for holidays and weekends).

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The m log-returns series $\{Z_{ij}\}_{j=1}^{n}$, are defined as:

$$Z_{ij} = \log \frac{S^i_{t_j+h}}{S^i_{t_j}} = \log S^i_{t_j+h} - \log S^i_{tj} \text{ ; for } j = 1,\ldots,m.$$

Assuming complete data, the MLE of μ and $\sigma\sigma'$ satisfy the equations:

$$\hat{\mu}_Z = (\hat{\mu} - \frac{1}{2}\operatorname{diag}(\widehat{\sigma\sigma'}))h \quad \text{and} \quad \hat{\Sigma}_Z = \widehat{\sigma\sigma'}h$$

where $\hat{\mu}_Z$ and $\hat{\Sigma}_Z$ are the sample mean vector and the covariance matrix of the series of the log-returns respectively. Thus,

$$\hat{\mu}^i_Z = \frac{1}{n}\Sigma^n_{j=1} Z_{ij} = \frac{1}{n}\Sigma^n_{j=1} \log \frac{S^i_{t_j+h}}{S^i_{t_j}} = \frac{1}{n}\log \frac{S^i_T}{S^i_0},$$

and

$$(\hat{\Sigma}_Z)_{ij} = \widehat{\operatorname{cov}}(Z^i, Z^j) = \frac{1}{n}\sum_{k=1}^{n}(Z_{ik} - \hat{\mu}^i_Z)(Z_{jk} - \hat{\mu}^j_Z).$$

Substituting, the parameters estimators are determined as:

$$\hat{\mu} = \frac{1}{h}\hat{\mu}_Z + \frac{1}{2}\operatorname{diag}(\widehat{\sigma\sigma'}) \quad \text{and} \quad \widehat{\sigma\sigma'} = \frac{1}{h}\hat{\Sigma}_Z .$$

In particular,

$$\hat{\mu}_i = \frac{1}{h}\hat{\mu}^i_Z + \frac{1}{2}\hat{\sigma}^2_i \quad \text{and} \quad \hat{\sigma}^2_i = (\widehat{\sigma\sigma'})_{ii} = \frac{1}{h}(\hat{\Sigma}_Z)_{ii}. \tag{3}$$

For more details, see Castañeda-Leyva *et al.* (2008).

2.1.2. Incomplete data estimation

Usually, the stock prices database has missing values. This may happen due to lack of market liquidity or because stock exchange authorities decide to stop trading the stock due to irregularities or anomalies in financial information, dim trading or problems between shareholders or management that may affect the company's stock pricing (Bolsa Mexicana de Valores, 2012).

A common, but non-desirable practice, is to eliminate the rows of the database corresponding to the days with at least one missing data of the m assets. As a consequence, the sample size decreases perturbing the estimations for the returns and volatilities from the m assets. Fortunately, the likelihood approach does not depend on whether the database is complete or not (Casella, 2002). The estimator is the same but the efficiency of the estimator decreases.

Let i be an asset that is observed in (perhaps) non equidistant times

$$0 = t_0 < t_1 < \cdots < t_n = T.$$

These times are not necessarily the same observed times as other asset prices given the asymmetry of incomplete data.

The maximum likelihood estimators for the rate of return μ_i and volatility σ_i are

$$\hat{\mu}_i = \frac{1}{h}\hat{\mu}_Z^i + \frac{1}{2}\hat{\sigma}_i^2 \quad \text{and} \quad \hat{\sigma}_i^2 = (\widehat{\sigma\sigma'})_{ii} = \frac{1}{h}(\hat{\Sigma}_Z)_{ii}, \tag{4}$$

where

$$\hat{\mu}_Z^i = \frac{1}{T}\sum_{j=1}^{n} Z_{ij} = \frac{1}{T}\log\frac{S_T^i}{S_0^i}$$

$$(\hat{\Sigma}_Z)_{ii} = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_j}\left(Z_{ij} - \frac{h_j}{h}\hat{\mu}_Z^i\right)^2.$$

The mathematical demonstration is provided in the Appendix. Equations (3) and (4) are the same cases where there are no missing data, i.e. $h_j = h$.

2.2. Calibration of the interest rate

A model for interest rates with a mean reverting trend is considered. The Vasicek model is convenient since this one factor and continuous time model is one of the most widely known and used in explaining the dynamics of interest rates (Cvitanic and Zapatero, 2004).

It is assumed that the dynamics of a zero-coupon bond $\{B_t\}_{t\geq 0}$, is given by the differential equation (DE)

$$dB_t = rB_t dt; \quad t \geq 0, \tag{5}$$

where $r > 0$ is the constant interest rate. The solution of this equation is the exponential function $B_t = B_0 e^{rt}$; $t \geq 0$. In order to calibrate this parameter, the time series of the Mexican treasury bill with 91 days maturity is used. This is the most marketable bond in the money market in Mexico.

The stochastic process $\{Y_t\}_{t\geq 0}$ for the interest rate satisfies the SDE

$$dY_t = -\lambda(Y_t - \mu)dt + \sigma dW_t, \; t > 0, \tag{6}$$

with $\lambda > 0$, $\mu \in \mathcal{R}$, $\sigma > 0$ and initial value $Y_0 = y \in \mathcal{R}$.

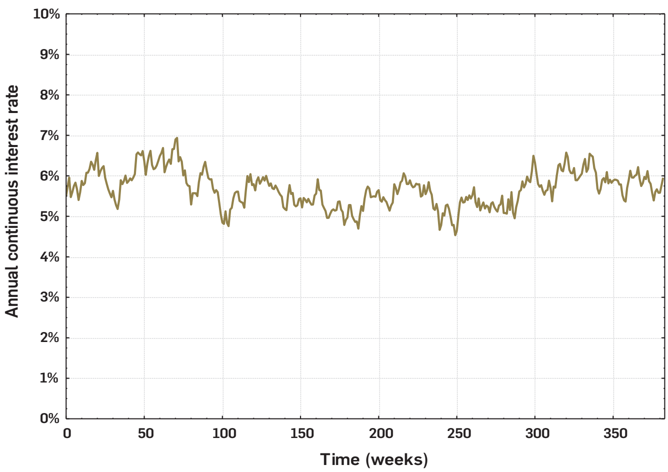
The parameter μ is named the long term mean or the asymptotic mean for the process. The parameter λ modulates the mean reversion rate. Mean reverting is a property that the interest rates display in the real financial markets. The volatility is represented by σ. Here $Y_0 = y$ is the value for the process at the initial time, which is assumed to be known.

The solution of (6) turns out to be (see Castañeda-Leyva *et al.*, 2008)

$$Y_t = e^{-\lambda t} y + (1-e^{-\lambda t})\mu + \sigma \int_0^t e^{-\lambda(t-s)}\, dW_s. \tag{7}$$

A simulation of the dynamics of the interest rate is shown in Figure 2. The reversion to the mean can be noticed as interest rates vary within a band.

Figure 2. Simulation of the Mexican Treasury bill dynamics using the Vasicek model with estimated historic parameters



SOURCE: WEEKLY DATABASE FROM BANCO DE MÉXICO (2010). 01/02/2003 TO 29/04/2010 DATA OF MEXICAN TREASURY BILLS WITH 91 DAYS MATURITY IS USED TO ESTIMATE PARAMETERS ACCORDING TO EQUATION (9)

Similar to model (1), the vector of parameters (λ, μ, σ) of model (6) is also estimated by MLE. In this sense, it is assumed that the process of interest rate $\{Y_t\}_{t\geq 0}$ is observed at times $0 = t_0 < t_1 < \cdots < t_n = T$, of the interval $[0, T]$, with $t_i = h_i$; for $i = 1,\ldots,n,$ and small $h = T/n > 0$. Here it is considered that the observed data are equidistant, that is, there are no missing data.

Expression (7) implies that, for all h, $t > 0$:

$$Y_{t+h} = (1-e^{-\lambda h})\mu + e^{-\lambda h} Y_t + \sigma \int_t^{t+h} e^{-\lambda s}\, dW_s. \tag{8}$$

The associated time series of interest rate $\{R_i\}_{i=0}^{n}$ holds

$$R_i = \beta_0 + \beta_1 R_{i-1} + \sigma_\varepsilon \varepsilon_i\,,\ i = 1,\ldots,n, \tag{9}$$

where $R_i = Y_{hi}$; for $i=0,1,\dots,n$, $\{\varepsilon_i\}_{i=1}^{n}$ is a Gaussian white noise, and

$$\beta_0 = \mu\,(1-\beta_1), \quad \beta_1 = e^{-\lambda h}, \quad \sigma_\varepsilon^2 = \frac{1}{2\lambda}\,(1-\beta_1^2)\sigma^2.$$

Equation (9) is a linear model and parameters $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\sigma}_\varepsilon$ can be obtained via least squares estimation or, equivalently, with MLE. Thus, the MLE for the parameters of the interest rate model (9) are

$$\hat{\mu} = \frac{\hat{\beta}_0}{1-\hat{\beta}_1}, \qquad \hat{\lambda} = -\frac{1}{h}\log\hat{\beta}_1, \qquad \hat{\sigma}^2 = \frac{2\hat{\lambda}}{1-\hat{\beta}_1^2}\,\hat{\sigma}_\varepsilon^2. \tag{10}$$

2.3. Wealth process

In this section the dynamics for the wealth process for the investor that follows a self-financing trading strategy is explained. Financial markets are composed by m risky assets and a risk-free bond that satisfy the SDE (1) and the DE (5) respectively. It is assumed that the vector of instantaneous rate of return μ and the symmetric matrix $\sigma\sigma$', along with the interest rate r, are known constants.

The investor's objective is to maximize wealth at the final time $T>0$. At every time t in the interval $[0,T]$, the investor decides the trading strategy, weighting and allocating each financial asset; that is, rebalancing the portfolio.

With an initial capital $X_0=x>0$, the wealth process $\{X_t\}_{0\le t\le T}$ is the sum of the investments in the risky assets and the bond. If $\pi_t^1, \dots, \pi_t^m$ denote the proportions of wealth that are invested in the m risky assets, then the investor allocates $\pi_t^i X_t$ amount of money to the $i-th$ risky asset. The residual money is invested in the bond, $\pi_t^0 X_t$ and total wealth is

$$X_t = \pi_t^0 X_t + \pi_t^1 X_t + \cdots + \pi_t^m X_t\,,$$

where the portfolio strategy $\pi_t = (\pi_t^1, \dots, \pi_t^m)'$; $t \in [0,T]$, satisfies the self-financing condition:

$$\pi_t^0 + \pi_t^1 + \cdots + \pi_t^m = 1.$$

This means that during the interval $[0,T]$, the capital gained is reinvested only in the two available financial securities. Under this assumption, the investor is not allowed to withdraw or consume funds from the portfolio or to make capital injections during the investment period. Though this condition seems restrictive to the convenience and sound decision making process of the investor, it is an important assumption that helps to quantify and make comparisons among portfolio strategies.

The wealth process $\{X_t\}_{0\le t\le T}$ and the trading strategy process $\{\pi_t\}_{0\le t\le T}$ are stochastic because they depend on the changing conditions of the market, but the portfolio is determined by the investor.

The portfolio strategy $\{\pi_t\}_{0\le t\le T}$ is admissible if the corresponding wealth process is non-negative, i.e. $X_t \ge 0$, $0 \le t \le T$. Should the wealth process vanish, say at time t, then from this moment the process stops and any type of investment and the wealth process becomes zero. Admissible portfolios allow short selling ($\pi_t^i < 0$) or borrowing ($\pi_t^0 < 0$). A final consideration for the financial market is the absence of taxes and spreads between saving and lending rates.

The wealth process can be described by the SDE

$$dX_t = \pi_t^0 X_t \frac{dB_t}{B_t} + \pi_t^1 X_t \frac{dS_t^1}{S_t^1} + \cdots + \pi_t^m X_t \frac{dS_t^m}{S_t^m} .$$

Substituting equations (1) and (5), the explicit solution for the wealth process is

$$X_t = X_0 e^{rt+((\mu-r1_m)\pi_t-\frac{1}{2}\pi_t'\sigma\sigma'\pi_t)t+\pi_t'\sigma W_t} ; \quad t \ge 0 . \tag{11}$$

Here 1_m is the vector of dimension m whose elements are ones. In particular, for deterministic and constant control, i.e., $\pi_t = \pi \in \mathcal{R}^m$, for $0 \le t \le T$ the solution of equation (11) becomes

$$X_t = X_0 e^{(r+(\mu-r1_m)'\pi-\frac{1}{2}\pi'\sigma\sigma'\pi)t+\pi'\sigma W_t} . \tag{12}$$

The investor will determine his portfolio strategy maximizing his utility function.

2.4. Portfolio optimization under stochastic control

The risk preferences of the investors are represented by a utility function $U(x)$, that is a real function strictly increasing and strictly concave. The investor's problem is to maximize the final expected utility wealth $EU(X_T)$. This study considers the cases for power and logarithmic utility function, defined as

$$U(x) = \begin{cases} \frac{1}{\gamma}x^\gamma, & x > 0, \quad 0 \ne \gamma < 1 \\ \log x, & x > 0, \quad \gamma = 0, \end{cases}$$

where γ is the risk propensity coefficient. If $\gamma \ll 0$, the investor is risk averse whereas the risk-taker preference is when $\gamma \approx 1$. Thus, the investor's problem is to maximize

$$J(\pi, x) = EU(X_T | X_0 = x) = E\left\{\frac{1}{\gamma} X_T^\gamma\right\},$$

where $\{\pi_t\}_{0\le t\le T}$ is an admissible trading portfolio and $X_0 = x > 0$ is the initial wealth.

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Applying the dynamic programming principle, this optimal problem is reduced to solving the associated Hamilton Jacobi Bellman equation (Fleming and Soner, 2006)

$$w_t + \sup_{\pi \in \mathrm{R}^m} \{x w_x [r + \pi'(\mu - r 1_m)] + \tfrac{1}{2} x^2 w_{xx} \pi' \sigma\sigma' \pi\} = 0 \, .$$

Here w_t, w_x and w_{xx} denote the corresponding derivatives of the function $w(t, x)$ with respect to t and x. Also, the supremum is taken for all the vectors π in $\mathcal{R}^m$. This new problem is now translated to one that does not depend on the state x, nor the time t:

$$\text{minimize} \left\{\tfrac{1}{2}\pi'\sigma\sigma'\pi + \pi'(\mu - r1_m)\right\}; \quad \text{for} \quad \pi \in \mathcal{R}^m \, .$$

This implies that the optimal investment is given for the deterministic constant portfolio strategy

$$\hat{\pi}(t, x) = \frac{1}{1-\gamma}(\sigma\sigma')^{-1}(\mu - r\,1_m); \;\; 0 \le t \le T \, . \tag{13}$$

This optimal strategy implies, at every moment, a constant assignment of wealth to each risky asset. The optimal strategy favors those assets with higher returns, while the risk aversion coefficient $1-\gamma$ and the uncertainty, represented by the square matrix $\sigma\sigma'$, inhibit investments in risky assets, channeling wealth to the risk-free bond.

Solution (13) summarizes the risk preferences along with the parameters of the market: r, μ, and $\sigma\sigma'$, being $\sigma\sigma' = \sigma^2$ if $m = 1$. Thus, the optimal portfolio will hold primarily risky assets if $\mu \gg r$, $\sigma \approx 0$, or $\gamma \approx 1$, which is in accordance with a rational risk-taker investor. Instead, the investment in the risky asset is discouraged, increasing investment in the riskless bond when $\mu \le r$, $\sigma \gg 0$, or $\gamma \ll 1$. Furthermore, the model short-sells the stock in the unusual case when $\mu < r$.

3. Data and parameter estimation

3.1. BSM model: Data and estimation

Daily stock prices are obtained from Infosel, a Mexican database that is widely used by stock brokers. An initial data screening is performed based on visual analysis of charts in order to eliminate stocks with low or minimum market liquidity. Historic daily, weekly and monthly quotes over a period 20 years for active firms are inspected. From this initial analysis, 54 stocks are selected, out of 247, the total number of stocks listed on the MSE during the analyzed period. Stocks are deleted because of low liquidity and long periods of missing quotations.

A second, more detailed scrutiny of the data, which includes the analysis of daily trading volume, reduces the sample to 40 stocks and the IPC index. The total number of observations is 1,843 prices for each stock and the index, from 02/01/03 to 04/23/10. This period is interesting in itself in terms of modeling prices because it encompasses moments of relative calm and also high instability in the global economy.

For comparison purposes, stock prices are scaled to the same initial relative price of 1, as of 01/02/2003 (the day of the first observation in the data sample), and those stocks with a missing price in the first observation are scaled to the same relative price of 28.76, the highest of the sample as of 04/23/2010 (the day of the last observation).

Stocks are classified regarding their data continuity. 20 stocks and the IPC index have complete data over the analyzed time interval, while the other stocks have some missing daily prices. Table 1 shows the details.

Table 1. Selected stocks with complete or missing data

	Ticker	Daily Prices
1	ALFA	Complete
2	ALSEA	Missing
3	AMX	Complete
4	ARA	Complete
5	ARCA	Missing
6	ASUR	Missing
7	AXTEL	Missing
8	BIMBO	Complete
9	CEMEX	Complete
10	COMERCI	Missing
11	CONTAL	Missing
12	ELEKTRA	Complete
13	FEMSA	Complete
14	GAP	Complete
15	GCARSO	Complete
16	GEO	Complete
17	GFAMSA	Missing
18	GFINBUR	Complete
19	GFNORTE	Complete
20	GMEXICO	Complete
21	GMODELO	Complete
22	GRUMA	Missing
23	HOGAR	Missing
24	HOMEX	Missing
25	ICA	Complete
26	ICH	Missing
27	IDEAL	Missing
28	KIMBER	Complete
29	KOF	Missing
30	PE	Missing
31	SARE	Missing
32	SIMEC	Missing
33	SORIANA	Missing
34	TELECOM	Complete
35	TELMEX	Complete
36	TLEVISA	Complete
37	TVAZTCA	Complete
38	URBI	Missing
39	VITRO	Missing
40	WALMEX	Complete

SOURCE: DAILY STOCK PRICES OF 40 STOCKS SELECTED FROM INFOSEL DATABASE FROM 01/02/2003 TO 04/23/2010

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In order to ensure the quality of the data, prices of selected stocks were compared with prices from a different database, Economatica – a subscription paid database for publicity traded firms from Latin America. This comparison was done especially for the stocks with missing data.

Parameters μ and σ, return and volatility, are estimated according to equations (3) and (4), for complete and incomplete data. Returns are estimated using log-returns Z_i, as indicated in section 2.1.1.

3.2. Robustness analysis of estimated parameter $\hat{\mu}$

A dynamic interval analysis is performed in order to have an empirical proof of the quality of the estimated parameter. The methodology consists of calculating, using the model estimator, daily annual rates of return $\hat{\mu}_i$, $i \in \{0,1, \ldots, n\}$, from the first price observation, going forward in time to the last observation $\hat{\mu}_i^F$, and going backward from the last to the first observation $\hat{\mu}_i^B$, and weighting $\hat{\mu}_i^F$ and $\hat{\mu}_i^B$ with a fixed index $i \in \{0,1, \ldots, n\}$, to propose a weighted estimator according to the formula:

$$\hat{\mu}_i = \delta\hat{\mu}_i^F + (1-\delta)\hat{\mu}_i^B;\ 0 \leq \delta \leq 1.$$

Two particular cases of this estimator are $\hat{\mu}_i = \hat{\mu}_i^B$ if $\delta = 0$ and $\hat{\mu}_i = \hat{\mu}_i^F$, for $\delta = 1$.

The comparison of the dynamic weighted estimator and complete interval estimator $\hat{\mu}$ is shown in Figures 3a and 3b for two stocks selected randomly, ALFA and WALMEX. In both figures, variations with respect to the dynamic interval weighted estimator are minimum, showing an error of no more than 0.2%, which demonstrates that estimator $\hat{\mu}$ is robust.

Figure 3a. **Dynamic weighted estimator and model estimator**

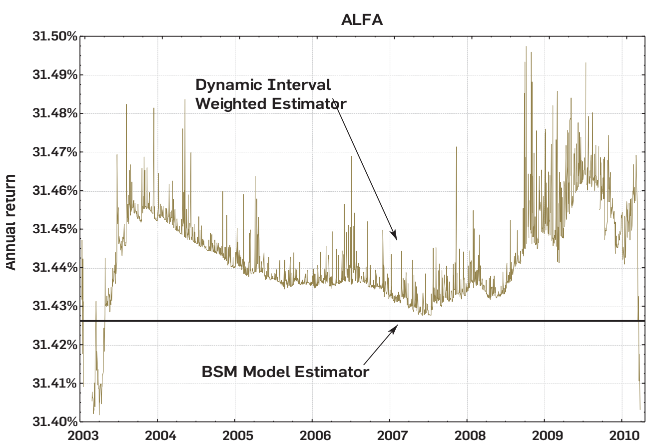

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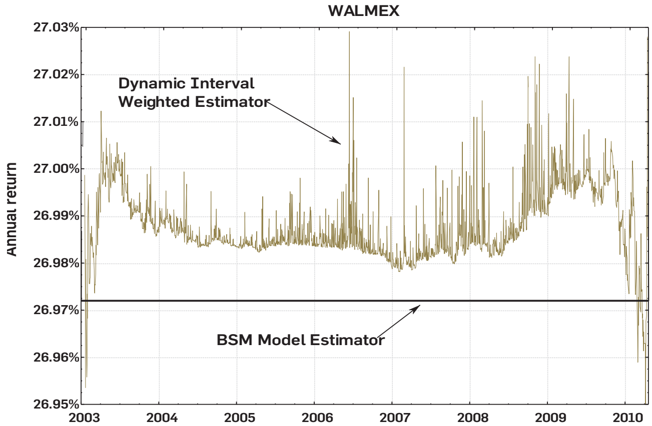
Figure 3b. **Dynamic weighted estimator and model estimator**



SOURCE: LOG-RETURNS ESTIMATED FROM DAILY CLOSING PRICES FROM INFOSEL FOR THE 01/02/2003 - 04/23/2010 PERIOD TO ESTIMATE PARAMETER $\hat{\mu}$

It is important to mention that $\hat{\mu}_i$ is an independent estimator, given that it represents the weighting of two of independent estimators, calculated at different time intervals. $\hat{\mu}_i$ is also a consistent estimator since it converges to the estimator $\hat{\mu}$. The determination of the index i and the weight factor δ, along with other statistical properties of the weighted estimator, are considered for future research.

3.3. Vasicek model: Data and estimation

Regarding interest rates, CETE's nominal returns are from databases by the Mexican Central Bank (Banco de México, 2010) from 02/01/03 to 04/23/10. The CETE is a widely used reference rate in México. Banco de México offers the CETE's through a closed auction every week. Issues are usually for maturities of 91, 182 and 364 days. In this study, 91-day maturity is used since its historic behavior shows the highest return with relative low volatility (Sáinz-Fernández, 2012). Time series of nominal returns of CETE are adjusted to instantaneous rates of return using the equivalence relation of nominal to continuous rates as follows,

$$e^r = \left(1+\frac{R_n}{n}\right)^n, \quad r = n \log\left(1+\frac{R_n}{n}\right).$$

The estimation of parameters β_0, β_1, and σ_ε of equation (9) is implemented using least squares regression and the estimation of the instantaneous interest rate, the rate of reversion to the mean and the volatility, $\hat{\mu}$, $\hat{\lambda}$ and $\hat{\sigma}$ respectively, are calculated from equations (10).

4. Results

4.1. Parameters of BSM model for stock prices

The estimates of μ and σ are shown in Table 2 and plotted in Figure 4 along with the regression line.

Table 2. **Return and volatility of 40 stocks and the market index**

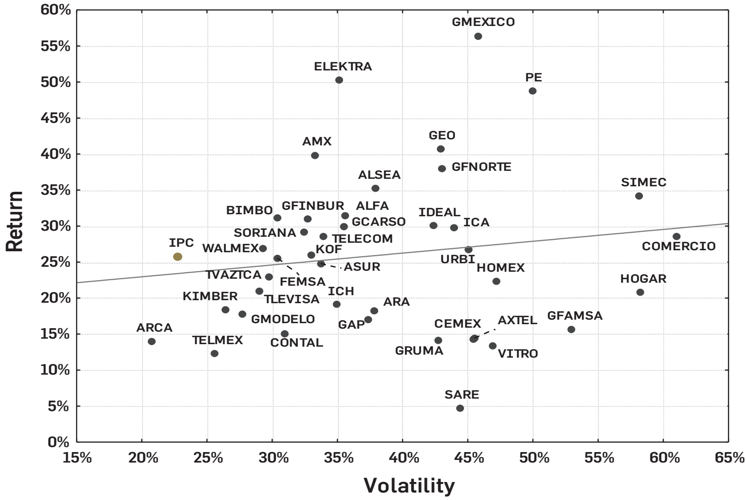
Stock		$\hat{\mu}$	$\hat{\sigma}$	Stock		$\hat{\mu}$	$\hat{\sigma}$
20	GMEXICO	56.4%	45.8%	17	FEMSA	25.6%	30.4%
12	ELEKTRA	50.3%	35.2%	6	ASUR	24.8%	33.7%
30	PE	48.8%	50.0%	37	TVAZTCA	22.9%	29.7%
16	GEO	40.7%	43.0%	24	HOMEX	22.3%	47.2%
3	AMX	39.9%	33.2%	36	TLEVISA	21.0%	29.0%
19	GFNORTE	38.0%	43.0%	23	HOGAR	20.8%	58.2%
2	ALSEA	35.3%	37.9%	26	ICH	19.1%	35.0%
32	SIMEC	34.2%	58.1%	28	KIMBER	18.4%	26.4%
1	ALFA	31.4%	35.6%	4	ARA	18.2%	37.8%
8	BIMBO	31.2%	30.4%	21	GMODELO	17.8%	27.7%
18	GINBUR	30.9%	32.7%	14	GAP	17.0%	37.3%
27	IDEAL	30.0%	42.4%	17	GFAMSA	15.7%	52.9%
15	GCARSO	29.9%	35.6%	11	CONTAL	15.1%	31.0%
25	ICA	29.8%	44.0%	7	AXTEL	14.5%	45.5%
33	SORIANA	29.2%	32.4%	9	CEMEX	14.3%	45.4%
34	TELECOM	28.6%	34.0%	22	GRUMA	14.2%	42.7%
10	COMERCI	28.5%	61.0%	5	ARCA	14.0%	20.8%
40	WALMEX	27.0%	29.3%	39	VITRO	13.3%	47.0%
38	URBI	26.7%	45.0%	35	TELMEX	12.3%	25.6%
29	KOF	26.0%	33.0%	31	SARE	4.7%	44.4%
	IPC	25.8%	22.8%	**Average**		26.0%	38.3%

SOURCE: LOG-RETURNS ESTIMATED FROM DAILY CLOSING PRICES OF 40 STOCKS AND THE MARKET (IPC) INDEX (INFOSEL DATABASE FROM 01/02/2003 TO 04/23/2010)

Some stocks perform atypically (outliers) like ALSEA, AMX, and ELEKTRA, that show high return and medium volatility; and GMEXICO, PE, GEO, and GFNORTE, which yield even higher returns. Other stocks like SARE, GRUMA, CEMEX, AXTEL, VITRO, and GFAMSA perform poorly and have high volatility, which make them discrepant points. SIMEC, COMERCI, and HOGAR have medium and low returns and extremely high volatility. The empirical results of the other stocks fall within the data cloud surrounding the regression line, validating the return-volatility theoretical relationship.

The IPC index yields average return and low volatility, which demonstrates the effect of diversification, embedded in market indexes, to reduce risk.

Figure 4. Return vs. volatility of 40 stocks and the market index



SOURCE: DATA FROM TABLE 2

4.2. Parameters of Vasicek model for the interest rate

The estimates of the parameters are shown in Table 3. The estimate of λ_r, the rate of reversion to the mean, is calculated using $h = 7/365.25$.

Table 3. Estimates of the parameters of the Vasicek model for interest rate

	Instantaneous rate	Nominal rate	Days
$\hat{\mu}_r$	5.55%	5.70%	
$\hat{\sigma}_r$	1.43%		
$\hat{\lambda}_r$			174.43

SOURCE: WEEKLY DATA BY BANCO DE MÉXICO (2010) FROM 01/02/2003 TO 04/29/2010

4.3. Results of the multivariate portfolio optimization

To compare optimal portfolios with the IPC index, the sample of stocks held in the portfolios is taken from the index (33 stocks) as of 07/16/2010 (Bolsa Mexicana de Valores, 2010). Since some new stocks in the index have less than four years trading, two of these stocks are eliminated, reducing the sample size to 31. Time series interval for the portfolio analysis is accordingly limited to four years, from 04/18/2006 to 04/23/2010. This time span includes both periods of stability and of volatility. Table 4 shows the selected stocks including weights and the estimates of the parameters, and in Figure 5 return and volatility for the selected stocks and the IPC index are plotted.

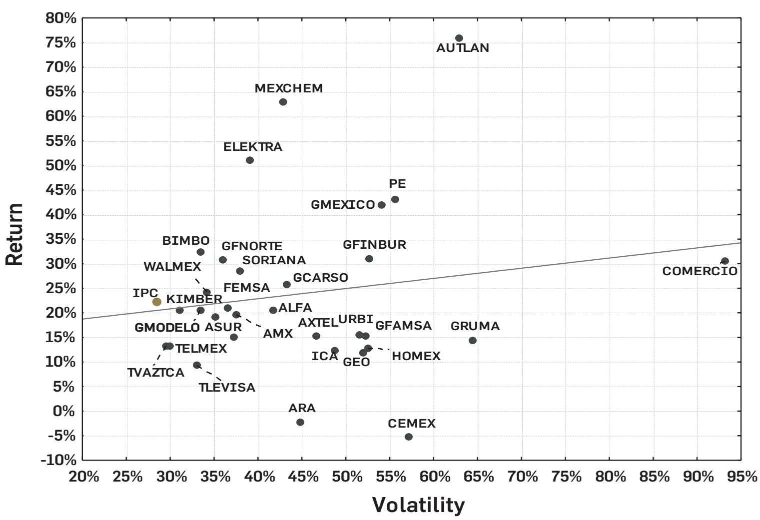
Table 4. Weights (π), $\hat{\mu}$ and $\hat{\sigma}$ of 31 stocks composing the IPC index

Stock	π	$\hat{\mu}$	$\hat{\sigma}$
1 ALFA	2.9%	20.5%	41.7%
2 AMX	26.9%	19.6%	37.5%
3 ARA	0.4%	-2.2%	44.8%
4 ASUR	1.0%	19.1%	35.1%
5 AUTLÁN	0.1%	75.9%	62.9%
6 AXTEL	0.5%	15.2%	46.6%
7 BIMBO	2.6%	32.3%	33.5%
8 CEMEX	6.6%	-5.3%	57.1%
9 COMERCI	0.3%	30.5%	93.2%
10 ELEKTRA	2.9%	51.0%	39.0%
11 FEMSA	5.4%	21.0%	36.5%
12 GAP	1.1%	15.0%	37.3%
13 GCARSO	1.8%	25.7%	43.3%
14 GEO	1.0%	12.0%	51.9%
15 GFAMSA	0.2%	15.3%	52.3%
16 GFINBUR	2.6%	30.8%	36.0%
17 GFNORTE	4.2%	31.1%	52.6%

Stock	π	$\hat{\mu}$	$\hat{\sigma}$
18 GMEXICO	6.9%	42.0%	54.0%
19 GMODELO	2.3%	20.6%	33.4%
20 GRUMA	0.3%	14.5%	64.4%
21 HOMEX	0.8%	12.7%	52.5%
22 ICA	1.1%	12.3%	48.7%
23 KIMBER	2.3%	20.6%	31.1%
24 MEXCHEM	1.4%	62.9%	42.9%
25 PE	1.8%	43.0%	55.6%
26 SORIANA	0.5%	28.6%	38.0%
27 TELMEX	2.5%	13.3%	30.0%
28 TLEVISA	6.5%	9.3%	33.0%
29 TVAZTCA	0.4%	13.3%	29.5%
30 URBI	0.7%	15.5%	51.5%
31 WALMEX	11.9%	24.3%	34.2%
IPC	100.0%	22.2%	28.6%
	Average	23.9%	45.2%

SOURCE: LOG-RETURNS ESTIMATED WITH DAILY CLOSING PRICES OF 31 STOCKS (INFOSEL DATABASE FROM 429 04/18/2006 TO 04/23/2010) AND THE IPC INDEX COMPOSITION AS OF 07/16/2010 (BOLSA MEXICANA DE 430 VALORES, 2010). THE "AVERAGE" IS ESTIMATED WITHOUT THE IPC INDEX PARAMETERS.

Figure 5. Return vs. volatility of 31 stocks composing the sample of the IPC index



SOURCE: DATA FROM TABLE 4

4.3.1. Market Portfolio and the Capital Market Line assuming short selling

The Market Portfolio is calculated with the BSM model, assuming the estimated risk free rate r = 5.55% (Table 3). The weights and parameters of the Market Portfolio are shown in Table 5, ranked from the highest to the lowest stock weights. The Capital Market Line is plotted in Figure 6, using coordinates calculated in Table 6 for a range of proportions of the Market Portfolio (π) from 0% to 130%. Bold figures in Table 6 and Table 7 show the Market Portfolio parameters.

Table 5. **Weights (π), returns ($\hat{\mu}$) and volatilities ($\hat{\sigma}$) of stocks composing the Market Portfolio allowing for short selling**

Stock		π	$\hat{\mu}$	$\hat{\sigma}$	Stock		π	$\hat{\mu}$	$\hat{\sigma}$
24	MEXCHEM	73.4%	62.9%	42.9%	14	GEO	1.4%	12.0%	51.9%
10	ELEKTRA	46.2%	51.0%	39.0%	20	GRUMA	-2.0%	14.5%	64.4%
7	BIMBO	24.0%	32.3%	33.5%	12	GAP	-2.6%	15.0%	37.3%
5	AUTLAN	21.4%	75.9%	62.9%	1	ALFA	-3.9%	20.5%	41.7%
23	KIMBER	21.1%	20.6%	31.1%	13	GCARSO	-7.9%	25.7%	43.3%
2	AMX	17.7%	19.6%	37.5%	30	URBI	-8.2%	15.5%	51.5%
18	GMEXICO	16.8%	42.0%	54.0%	21	HOMEX	-8.3%	12.7%	52.5%
11	FEMSA	16.7%	21.0%	36.5%	6	AXTEL	-8.4%	15.2%	46.6%
31	WALMEX	14.8%	24.3%	34.2%	15	GFAMSA	-8.5%	15.3%	52.3%
17	GFNORTE	14.7%	30.8%	36.0%	29	TVAZTCA	-23.6%	13.3%	29.5%
25	PE	11.6%	43.0%	55.6%	28	TLEVISA	-30.1%	9.3%	33.0%
4	ASUR	10.3%	19.1%	35.1%	8	CEMEX	-30.9%	-5.3%	57.1%
19	GMODELO	8.9%	20.6%	33.4%	22	ICA	-35.0%	12.3%	48.7%
14	GFINBUR	8.6%	31.1%	52.6%	3	ARA	-49.80%	-2.20%	44.80%
9	COMERCI	7.8%	30.5%	93.2%	Portfolio parameters				
26	SORIANA	2.1%	28.6%	38.0%	$\hat{\mu}$			119.1%	
27	TELMEX	1.8%	13.3%	30.0%	$\hat{\sigma}$			44.5%	

SOURCE: LOG-RETURNS ESTIMATED WITH DAILY CLOSING PRICES OF 31 STOCKS TAKEN FROM INFOSEL DATABASE FROM 04/18/2006 TO 04/23/2010

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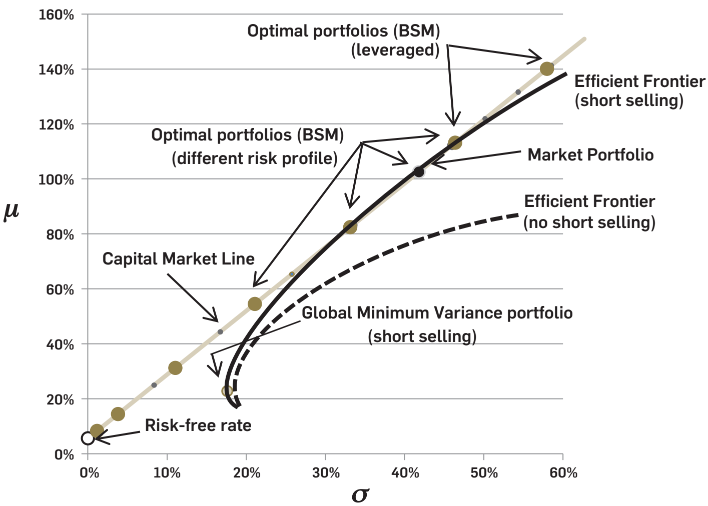
Table 6. **Portfolios estimated with different weights of the Market Portfolio (π) and the risk-free bond along the Capital Market Line**

π	100%–π	$\hat{\mu}$	$\hat{\sigma}$
0%	100%	5.5%	0%
20%	80%	28.3%	8.9%
40%	60%	51.0%	17.8%
60%	40%	73.7%	26.7%
80%	20%	96.4%	35.6%
90%	10%	107.8%	40.0%
100%	**0%**	**119.1%**	**44.5%**
110%	-10%	130.5%	49.0%
120%	-20%	141.8%	53.4%
130%	-30%	153.2%	57.8%

SOURCE: LOG-RETURNS ESTIMATED WITH DAILY CLOSING PRICES OF 31 STOCKS TAKEN FROM INFOSEL DATABASE FROM 04/18/2006 TO 04/23/2010 AND ESTIMATED PARAMETER OF INTEREST RATE

Figure 6. **Optimal portfolios allowing for short selling**



SOURCE: DATA FROM TABLES 5 AND 6

The Global Minimum Variance portfolio is calculated from the formula derived by Merton (1972) and is shown in Figure 6 for short selling and in Figure 7 for no short selling restriction.

4.3.2. Optimal portfolios-BSM model

Optimal portfolios are calculated for different values of the risk propensity coefficient γ. Portfolios are located along the Capital Market Line, depending on the value of γ.

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Return and volatility of the portfolios are shown in Table 7 for different values of γ. Some of these portfolios are indicated in Figure 6.

Table 7. Optimal portfolios for different risk profiles (γ) estimated with BSM model composed by the Market Portfolio (π) and risk-free bond

γ	π	100%–π	$\hat{\mu}$	$\hat{\sigma}$
0.1	637.6%	-537.6%	729.8%	283.7%
0	573.9%	-473.9%	657.3%	255.3%
-0.1	521.7%	-421.7%	598.1%	232.1%
-1	286.9%	-186.9%	331.4%	127.7%
-3	143.5%	-43.5%	168.5%	63.8%
-4	114.8%	-14.8%	135.9%	51.1%
-4.7	**100.0%**	**0.0%**	**119.1%**	**44.5%**
-5	95.6%	4.4%	114.2%	42.6%
-6	82.0%	18.0%	98.7%	36.5%
-10	52.2%	47.8%	64.8%	23.2%
-20	27.3%	72.7%	36.6%	12.2%
-60	9.4%	90.6%	16.2%	4.2%
-500	1.1%	98.9%	6.9%	0.5%

SOURCE: LOG-RETURNS ESTIMATED WITH DAILY CLOSING PRICES OF 31 STOCKS TAKEN FROM INFOSEL DATABASE FROM 04/18/2006 TO 04/23/2010 AND ESTIMATED PARAMETER OF INTEREST RATE

4.3.3. Comparison of optimal portfolios and the Mexican Stock Exchange index

Since the IPC is an index, optimal portfolios are calculated with positive weights, i.e. not allowing for short selling and assuming no investment in the risk-free bond. Table 8 shows the weights and parameters of the portfolio of the IPC index, the Global Minimum Variance Portfolio, three optimal portfolios, and the Market Portfolio, all with no short selling restriction. These portfolios are calculated by maximizing the Sharpe (1963) ratio, since the BSM optimal portfolio solution assumes short selling. The Sharpe ratio (θ) is also reported in Table 8 and parameter estimates $\hat{\mu}$ and $\hat{\sigma}$ for all portfolios are plotted in Figure 7, estimated from the BSM model.

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Table 8. Weights (π), returns ($\hat{\mu}$) and volatilities ($\hat{\sigma}$) for the IPC index, the Global Minimum Variance portfolio, three optimal portfolios and the Market Portfolio

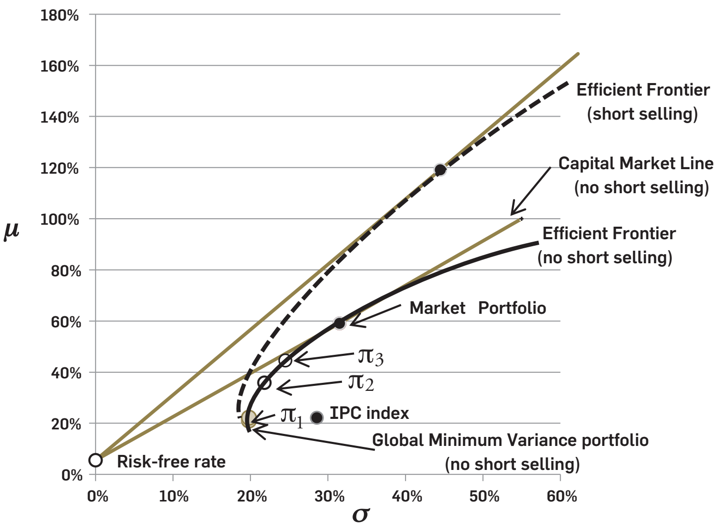
Stock	π_{IPC}	π_{GMV}	π_1	π_2	π_3	π_{MP}	$\hat{\mu}$	$\hat{\sigma}$
1 ALFA	2.9%	0.0%	0.0%	0.0%	0.0%	0.0%	20.5%	41.7%
2 AMX	26.9%	0.0%	0.0%	0.0%	0.0%	0.0%	19.6%	37.5%
3 ARA	0.4%	0.0%	0.0%	0.0%	0.0%	0.0%	-2.2%	44.8%
4 ASUR	1.0%	9.9%	9.9%	10.4%	8.3%	0.0%	19.1%	35.1%
5 AUTLAN	0.1%	0.0%	0.2%	6.1%	10.4%	19.3%	75.9%	62.9%
6 AXTEL	0.5%	0.0%	0.0%	0.0%	0.0%	0.0%	15.2%	46.6%
7 BIMBO	2.6%	0.0%	0.4%	4.4%	5.6%	0.6%	32.3%	33.5%
8 CEMEX	6.6%	0.0%	0.0%	0.0%	0.0%	0.0%	-5.3%	57.1%
9 COMERCI	0.3%	0.0%	0.0%	0.0%	0.0%	0.0%	30.5%	93.2%
10 ELEKTRA	2.9%	5.2%	6.4%	15.7%	21.7%	32.8%	51.0%	39.0%
11 FEMSA	5.4%	0.0%	0.0%	0.0%	0.0%	0.0%	21.0%	36.5%
12 GAP	1.1%	5.5%	5.0%	0.1%	0.0%	0.0%	15.0%	37.3%
13 GCARSO	1.8%	0.0%	0.0%	0.0%	0.0%	0.0%	25.7%	43.3%
14 GEO	1.0%	0.0%	0.0%	0.0%	0.0%	0.0%	12.0%	51.9%
15 GFAMSA	0.2%	0.0%	0.0%	0.0%	0.0%	0.0%	15.3%	52.3%
16 GFINBUR	2.6%	14.8%	14.8%	14.2%	13.0%	6.9%	30.8%	36.0%
17 GFNORTE	4.2%	0.0%	0.0%	0.0%	0.0%	0.0%	31.1%	52.6%
18 GMEXICO	6.9%	0.0%	0.0%	0.0%	0.0%	0.0%	42.0%	54.0%
19 GMODELO	2.3%	8.2%	8.2%	6.6%	3.6%	0.0%	20.6%	33.4%
20 GRUMA*	0.3%	4.7%	4.4%	0.7%	0.0%	0.0%	14.5%	64.4%
21 HOMEX	0.8%	0.0%	0.0%	0.0%	0.0%	0.0%	12.7%	52.5%
22 ICA	1.1%	0.0%	0.0%	0.0%	0.0%	0.0%	12.3%	48.7%
23 KIMBER	2.3%	17.2%	17.1%	15.1%	12.6%	0.0%	20.6%	31.1%
24 MEXCHEM	1.4%	0.0%	0.9%	14.4%	23.0%	40.4%	62.9%	42.9%
25 PE	1.8%	0.0%	0.0%	0.0%	0.0%	0.0%	43.0%	55.6%
26 SORIANA	0.5%	2.5%	2.4%	0.0%	0.0%	0.0%	28.6%	38.0%
27 TELMEX	2.5%	14.0%	13.2%	5.5%	0.0%	0.0%	13.3%	30.0%
28 TLEVISA	6.5%	0.0%	0.0%	0.0%	0.0%	0.0%	9.3%	33.0%
29 TVAZTCA	0.4%	15.2%	14.2%	3.7%	0.0%	0.0%	13.3%	29.5%
30 URBI	0.7%	0.0%	0.0%	0.0%	0.0%	0.0%	15.5%	51.5%
31 WALMEX	11.9%	2.8%	2.9%	3.0%	1.7%	0.0%	24.3%	34.2%

Portfolios parameters

	π_{IPC}	π_{GMV}	π_1	π_2	π_3	π_{MP}
$\hat{\mu}$	22.2%	21.1%	22.2%	36.0%	44.6%	59.1%
$\hat{\sigma}$	28.6%	19.8%	19.8%	21.8%	24.5%	31.5%
θ	0.58	0.78	0.84	1.40	1.59	1.7

SOURCE: LOG-RETURNS ESTIMATED WITH DAILY CLOSING PRICES OF 31 STOCKS TAKEN FROM INFOSEL DATABASE FROM 04/18/2006 TO 04/23/2010

Figure 7. **Optimal portfolios with no short selling and the IPC index**



SOURCE: DATA FROM TABLE 8

These results show that the IPC index performs poorly with respect to portfolios π_1, π_2, π_3 and the Market Portfolio. This is due to the relative weight of the stocks that compose the index since all portfolios hold the same stocks as the index.

5. Conclusions

The results of modeling prices of stocks listed on the MSE with the Black-Scholes-Merton (BSM) time continuous stochastic model are encouraging. The mathematical development based on MLE inference of the return and volatility parameters, for datasets with complete and with missing data, is an important empirical contribution of this study as the estimation of parameters for stock price time series frequently presents asymmetrical missing daily quotations. From the academic perspective, this study complements the limited literature related to modeling prices and optimal portfolios of Mexican stocks.

Using two databases widely utilized by MSE brokers, Infosel and Economatica, a meticulous exploratory analysis for the selection of stocks with price series of good quality was carried out. Stocks with low or no liquidity were eliminated and a detailed scrutiny and graphical analysis was done to detect discrepant data or missing quotations since the MSE is not considered a deep market. A sample of 40 stocks was selected for approximately seven years that included periods of both stability and high volatility.

Once the methodology of estimation was applied using the selected data series, the robustness of the estimation was verified by means of an analysis of dynamic intervals, comparing the return estimation with a dynamic interval weighted estimator. It is pertinent to mention that in this research the dynamic weighted estimator was only used for the purpose of contrasting the estimator of the rate of return of the model, and only the properties of independence and consistency of the dynamic estimator were mentioned, but it is suggested that these dynamic weighted estimators be considered for future research.

For the purpose of comparing optimal portfolios estimated with the BSM parameters, stocks were selected from the IPC index sample, including 31 stocks and a time interval of four years. The portfolios obtained with the BSM model using the power utility function and allowing for short selling were consistent with the investor's risk propensity coefficient. The optimization of portfolios using the power utility function to include the risk preferences of the investor in the model, which other models such as Markowitz (2012)[5] do not consider, is an empirical contribution of this research to portfolio theory applications in the MSE.

For the comparison of optimal portfolios with the IPC index portfolio, the restriction of no short selling was considered. The Global Minimum Variance portfolio, three optimal portfolios on the Efficient Frontier as well as the Market Portfolio were estimated. All portfolios outperformed the IPC index showing higher returns and lower volatility. It is important to emphasize that the need for the development of other methodologies of efficient representation of the index, different from the market capitalization of stocks, is an issue that could be the subject of deeper research.

The fact that estimated optimal portfolios outperformed the IPC index suggests that portfolio managers, who frequently recommend investments in funds that follow the behavior of the index (index tracker funds), might not be giving the best advice to their customers since other superior portfolios could be selected.

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[5] *Markowitz distinguishes three types of expected utility maximization: explicit (this paper approach), M-V approximate (his approach) and implicit (Levy and Markowitz, 1979).*

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Appendix

Parameters inference with missing data using MLE. Variable *h* approach

Assume the price process $\{S_t\}_{t\geq 0}$ is observed at time series

$$0 = t_0 < t_1 <, ..., < t_n = T.$$

If daily time series has no missing data, then

$$h = t_i - t_{i-1} = 1 \text{ day}.$$

It is important to mention that h is related to working days only so that holidays and weekends are not considered missing data days and it is also assumed that there exists one day step from Friday to Monday. The series of log-returns are

$$Z_i \triangleq \log \frac{S_t}{S_{t-1}} = \log S_t - \log S_{t-1} \; ; \; i = 1, ..., n.$$

According to the model, the distribution of Z_i is

$$Z_i \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)h, \sigma^2 h\right)$$

with

$$\mu_Z = (\mu - \frac{1}{2}\hat{\sigma}^2)h$$

and

$$h = t_i - t_{i-1} .$$

Note that now

$$\sum_{i=1}^{n} h_i = \sum_{i=1}^{n} (t_i - t_{i-1}) = T .$$

The joint probability density function of $(Z_1,\ldots, Z_n)$ is

$$f(Z_1,\ldots, Z_n) = \prod_{i=1}^{n} f(z_i) = \prod_{i=1}^{n} \left(\frac{1}{\sigma\sqrt{h_i}}\right)\varphi\left(\frac{z_i-\mu_z h_i}{\sigma\sqrt{h_i}}\right)$$

$$= \frac{1}{(2\pi)^{n/2}} \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{h_i}}\, e^{-\frac{1}{2\sigma^2 h_i}(z_i-\mu_z h_i)^2}$$

$$= \frac{1}{(2\pi)^{n/2}\prod_{i=1}^{n} h_i^{1/2}} \frac{1}{\sigma^n}\, e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\frac{1}{h_i}(z_i-\mu_z h_i)^2} \quad .$$

The likelihood and log-likelihood functions are,

$$L(\mu_z, \sigma) = \frac{1}{\sigma^n}\, e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\frac{1}{h_i}(z_i-\mu_z h_i)^2}$$

and

$$l(\mu_z, \sigma) = -n \log \sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n} \frac{1}{h_i}(z_i-\mu_z h_i)^2 \,.$$

Deriving with respect to μ_z and equating to zero,

$$\frac{\partial}{\partial M_z}\, l(\mu_z, \sigma) = \frac{2}{2\sigma^2}\sum_{i=1}^{n} \frac{1}{h_i}(z_i-\mu_z h_i) h_i \,,$$

$$0 = \sum_{i=1}^{n}(z_i-\mu_z h_i) = \left(\sum_{i=1}^{n} z_i - \mu_z \sum_{i=1}^{n} h_i\right) = \left(\sum_{i=1}^{n} z_i - \mu_z T\right),$$

clearing $\hat{\mu}_z$,

$$\hat{\mu}_z = \frac{1}{T}\sum_{i=1}^{n} z_i = \frac{1}{T}(\log S_T - \log S_0),$$

and for stock i :

$$\hat{\mu}_z^i = \frac{1}{T} \log \frac{S_T^i}{S_0^i} \,.$$

Deriving with respect to σ and equating to zero:

$$\frac{\partial}{\partial\sigma} l(\mu_z, \sigma) = -\frac{n}{\sigma} + \frac{2}{2\sigma^3}\sum_{i=1}^{n} \frac{1}{h_i}(z_i - \mu_z h_i) h_i$$

$$0 = \frac{n}{\sigma^3}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{(z_i - \mu_z h_i)^2}{h_i} - \sigma^2\right].$$

For the sample variance, clearing $\hat{\sigma}_z^2$,

$$\hat{\sigma}_z^2 = \frac{1}{n}\sum_{i=1}^{n}\frac{(z_i - \mu_z h_i)^2}{h_i},$$

and in particular for stock i:

$$(\hat{\Sigma}_z)_{ii} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_j}\left(z_{ij} - \frac{h_j}{h}\hat{\mu}_z^i\right)^2.$$